The gradient flow of the double well potential and its appearance in interacting particle systems

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Summary

In this work we are interested in the existence of solutions to parabolic partial differential equations associated to gradient flows which involve the so-called double well potential

$$\mathcal{F}(u) := \int V(u_x)\ dx \tag{0.1}$$

with

$$V(p) := \frac{1}{4}(p^2 - 1)^2. \tag{0.2}$$

The functional (0.1) is a nonconvex and nonconcave functional, and therefore the formal L^2-gradient flow of (0.1) leads to a so-called *forward-backward parabolic equation*, which reads

$$\begin{aligned} u_t &= (V'(u_x))_x \\ u(0) &= \overline{u}. \end{aligned} \tag{0.3}$$

Due to the non-convexity of (0.1), the partial differential equation (0.3) is not well-posed: it may fail to admit local in time classical solutions, at least for a large class of initial data $\overline{u}$.
Moreover, if the initial data takes values in the concave region of V, instabilities, such as the quick formation of microstructures, are expected, making the subsequent evolution difficult to describe.
A widely used remedy for this is to add a higher order term in the functional (0.1), for example $\frac{1}{2}u_x^2$, which leads to the Ginzburg–Landau energy

$$\mathcal{E}(u) := \int \frac{1}{2}u_x^2 + V(u)\, dx. \tag{0.4}$$

In contrast to double well potential $\mathcal{F}$ in (0.1), the regularized functional $\mathcal{E}$ in (0.4) is strictly convex in the higher derivative and the L^2-gradient flow of (0.4) is a well-posed partial differential equation, the so-called Allen-Cahn equation.

In this work we consider a stochastic perturbation of the L^2-gradient flow of (0.4). By this we mean the infinite dimensional diffusion

$$\partial_t u(x,t) \;=\; -\frac{\delta\mathcal{E}}{\delta u}(u)(x,t) \;+\; \sqrt{2\sigma}\frac{\partial^2}{\partial_x\partial_t}W(x,t) \tag{0.5}$$

where $\delta\mathcal{E}$ denotes the Fréchet differential of the potential $\mathcal{E}$ given by (0.4) and $\frac{\partial^2}{\partial_x\partial_t}W(x,t)$ denotes space-time white noise.
The associated stochastic partial differential equation (SPDE) to (0.5) is called the Stochastic Allen-Cahn equation and reads

$$\partial_t u(x,t) = \partial_x^2 u(x,t) - u^3(x,t) + u(x,t) + \sqrt{2\sigma}\frac{\partial^2}{\partial_x\partial_t}W(x,t). \tag{0.6}$$

The stochastic term in (0.6) reflects small scale noise that perturbs the deterministic dynamics. The noise gives rise to metastability: If the system starts at one local energy minimizer, then the solution will exhibit small fluctuations around this state until it quickly crosses the energy barrier and moves into a small neighborhood of the other minimizer.

Outline of this work

The first two chapters give a very brief overview on basic concepts and results on the equations studied in this work.
In Chapter 3 and Chapter 4 we prove the existence of solutions to (0.3) for sufficiently small initial data, using a discrete-in-space scheme.
In Chapters 5 -7 we also consider a discrete-in-space scheme, which can be interpreted as an interacting particle system with long-range interaction on a lattice $\Lambda = \mathbb{Z}/N\mathbb{Z}$. Each particle is subject to force derived from a bistable potential like (0.2) and perturbed by white noise.
In Chapter 5 we consider the case that in this system of N particles, a fixed number of particles R are interacting with each other. We show that after suitable rescaling, the particle system converges, as system size N goes to infinity, to a well-posed stochastic PDE.
In Chapter 6 we generalize the above result to the case when both the range of interaction R and the number of particles N simultaneously go to infinity. Putting suitable conditions on the weight, we can prove convergence to stochastic Allen-Cahn equation (0.6).
In Chapter 7 we look at metastability phenomena arising in the particle system of Chapter 6. While in the deterministic case there are exactly two stable states of the evolution law described by the Allen-

Cahn equation, which correspond to the two minima of the potential V, in the presence of noise, these states are merely *metastable*, meaning that it is possible that there is a switch from one minima to the other.
We show in Chapter 7 that the transition times of the discrete interacting particle system converge, as both the range of interaction R and the system size N go to infinity, to those of (0.6), for which Eyring-Kramers-type laws are known.

Chapter 1

Introduction

In this work, we study phenomena arising in physics with tools from applied analysis, probability theory and partial differential equations. In particular, we consider phenomena which are modeled by the minimisation of a certain energy. Setting the evolution in time as the steepest descent of the energy, we can associate a parabolic PDE to the energy functional. We call this the *gradient flow* of the energy, which motivates the title of this work.
One technique we will use is to approximate the gradient flow under consideration by a semidiscrete scheme, which can be called a "discrete in space evolution equation".

1.1 Discrete ill-posed nonlinear evolution equations

The first part of this work is devoted to the study of convergence of a so-called "discrete ill-posed nonlinear evolution equation", which looks as

$$\begin{aligned} \frac{du^h}{dt} &= D_h^+ \, W'(D_h^- u^h) && \text{in } \mathbb{T} \times [0, +\infty), \\ u^h(0) &= \overline{u}^h && \text{on } \mathbb{T}, \end{aligned} \tag{1.1}$$

where $\mathbb{T}$ is a one-dimensional torus, discretized with a grid of N subintervals of equal length h, $D_h^{\pm}$ are difference quotients, $\overline{u}^h$ are the piecewise linear discrete initial data and W is a double well potential, for example $W(p) := \frac{1}{4}(p^2 - 1)^2$.
Such discrete in space evolution equations arise for example in the study of formations of shear bands in granular flows [118], in popula-

tion dynamics or as a class of models of reinforced random walks on a lattice [73, 100] modeling the chemotactic movement of bacteria.
Moreover, they appear in computer vision as models of image denoising and segmentation [22, 23, 103], see also [9] for an overview.
The wording was chosen because these equations, though they are well-posed themselves, resemble finite difference approximations of ill-posed nonlinear diffusion equations, see for example [48]. The lack of well-posedness is often due to the fact that the continuum version of the equations has a variable direction of parabolicity, which is why they are called "forward-backward parabolic equations". We refer to reader to Chapter 2, Section 2.2 for a detailed explanation.
For certain applications, like digital images and granular media, which are "discrete by nature", it is sufficient to consider the continuous PDE only as a heuristic model and derive all properties essential for the application in the discrete setting, as performed for example in [48].
The viewpoint that we take in Chapter 3 and 4 is to consider the discrete scheme as a regularization of the ill-posed problem and to prove converge of solutions to a well-posed limit problem for certain initial data.

The L^2-gradient flow of the double well potential The forward-backward parabolic limit equation to our discrete scheme (1.1) reads

$$u_t = (W'(u_x))_x \qquad \text{in } \mathbb{T} \times [0, +\infty), \tag{1.2}$$

which is the formal L^2-gradient flow of the nonconvex and nonconcave functional

$$F(u) := \int_{\mathbb{T}} W(u_x)\, dx,$$

where $\mathbb{T}$ is the one-dimensional torus and W is a standard double-well potential as before.
As (1.2) is not well-posed due to the nonconvexity of W, it may fail to admit local in time classical solutions, at least for a large class of initial data. A typical source of instability is, for example, the case when there are intervals $I \subset \mathbb{T}$ for which the initial data takes values in the concave region of W. Indeed, under these conditions the backward character of the equation manifests and instabilities, such as the quick formation of microstructures, are expected, making the subsequent evolution difficult to describe.
Instead of a regularisation via semidiscretisation, as in our case or in [44], one might also study continuous regularisations of the ill-posed PDEs and try to pass to the limit. In fact, Ennio De Giorgi [41] conjectured the existence of a limit to a fourth-order parabolic

regularisation. This regularisation has been studied by Slemrod in [109], and by Bellettini, Fusco and Guglielmi in [14], where numerical evidence of the conjecture was given. A rigorous proof of DeGiorgi's conjecture is still missing. We give a short overview to these and other regularisations in Chapter 2.
A famous forward-backward parabolic equation, which has received quite some attention in the last fifteen years, is the Perona-Malik equation for image denoising. Here, several different continuous regularisation, see for example the regularizations in space [3, 13], or via time delay [11] have been studied. Other attempts to establish well-posedness was by defining appropriate notions of weak solutions as for example in [16].

Results presented in this work

In Chapter 3 we tackle the question on existence of solutions to (1.2) via approximation with the discrete in space initial boundary value problem (see (3.13))

$$\begin{aligned} \frac{du^h}{dt} &= (D_h^+ W'(D_h^- u^h))\,, && \text{in } [0,1]\times[0,T] \\ u^h(0,t) &= u^h(1,t) && \text{on } \{0,1\}\times[0,T] \\ D^- u^h(0,t) &= D^- u^h(1,t) && \text{on } \partial\{0,1\}\times[0,T] \\ u^h(.,0) &= \overline{u}^h && \text{on } [0,1]\times\{0\} \end{aligned} \tag{1.3}$$

where $\overline{u}^h$ is a piecewise linear one-periodic initial data with non-differentiable points $a_1^h, \ldots, a_m^h$.
The crucial estimate is Proposition 3.2.10, where we show that if the initial datum satisfies

$$\alpha \;\geq\; \left|D_h^- \overline{u}^h\right| \;\geq\; \frac{1}{\sqrt{3}} \quad \text{in each interval } [a_j^h h, a_{j+1}^h h] \tag{1.4}$$

with α such that $W'(\alpha) = W'(-\frac{1}{\sqrt{3}})$, then this property holds for all times t.
Our main result in Chapter 3 is Theorem 3.3.5, where we show that one can pass to the (full) limit in such solutions, as $h \to 0^+$. The limit function u is a continuous, piecewise smooth solution to the PDE

$$\begin{aligned} &(i) && u_t = W'(u_x)_x && \text{in } (\mathbb{T}\setminus J_{u_x})\times[0,T] \\ &(ii) && W'(u_x(a_j^+)) = W'(u_x(a_j^-)) && \text{on } J_{u_x}\times[0,T] \\ &(iii) && u(a_j^+) = u(a_j^-) && \text{on } J_{u_x}\times[0,T] \\ &(iv) && u(0) = \overline{u} && \text{at } \mathbb{T}\times\{0\} \end{aligned} \tag{1.5}$$

where $J_{u_x} := \{a_1, \dots, a_m\}$ and $\mathbb{T}$ denotes the interval $[0,1]$ equipped with periodic boundary conditions[1].
Note that the solution to (1.5) is such that $W'(u_x(\cdot,t))$ is continuous at the points where there is a jump from one connected component to another, namely where $u_x(\cdot,t)$ has a discontinuity. Moreover, due to Proposition 3.2.10, the solution at these points (where there is a jump from one connected component to another) does not move in the horizontal direction. This means in particular that no coarsening phenomena is observed for solutions to (1.5). We compare our result and the abovementioned properties of the obtained solution to similar results obtained by different regularisations of (1.2) in Chapter 2, Section 2.2.
In Theorem 3.3.9, we characterize the long-time behaviour of the solution: as $t \to \infty$, the gradient of the solution becomes a piecewise-constant function which assumes exactly two values, a positive and a negative one, which are lying in the stable region.

In Chapter 4 we show convergence of (4.6) for a larger class of initial data, namely those satisfying

$$|\overline{u}_x(x)| \leq 2/\sqrt{3}, \qquad x \in \mathbb{T}, \tag{1.6}$$

The idea here is to approximate the "bad" slopes $p \leq \frac{1}{\sqrt{3}}$ by suitable percentage of the corresponding "good" slopes p^- and p^+, satisfying the condition $W'(p) = W'(p^-) = W'(p^+)$. We encode this in the function $\overline{\varrho}$, which we call "piecewise constant rational percentage", see Definition 4.4.1. We refer to Section 4.3.2 for an example. In this way we can construct a sequence $(\overline{u}^h)$ of initial data (depending on $\overline{\varrho}$) and converging to $\overline{u}$ as $h \to 0^+$, with the crucial property that the discrete gradients $D_h^- \overline{u}^h$ always belong to $\{W'' \geq 0\}$ (see Section 4.4.2).

1.2 Interacting particle systems

In the second part of this work, we study another discrete in space evolution equation, this time with a stochastic term, so that the approximating scheme is not a system of ODEs, but a system of SDEs. Our approach in the stochastic case is motivated as being the procedure of taking a *scaling limit* to an *interacting particle system.*
Interacting particle systems model complex phenomena in natural and social sciences, such as traffic flow on highways or pedestrians, opinion dynamics, spread of epidemics or fires, reaction diffusion systems,

[1]The notation $\mathbb{T}$ was introduced during the process of this work, which is why it does not appear in Chapter 3. We apologize for the confusion.

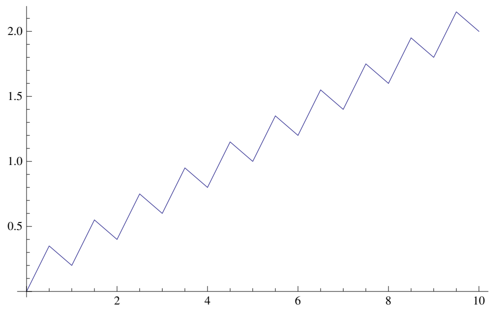

Figure 1.1: *approximation of slope* $p = 0.2$ *half slopes* $p^- \in [-\alpha, -\frac{1}{\sqrt{3}}]$ *and half* $p^+ \in [\frac{1}{\sqrt{3}}, \alpha]$, *i.e.* $\sigma = \overline{\varrho}(x) = 0.5$

crystal surface growth, chemotaxis and financial markets. These phenomena involve a large number of interacting components, which are modeled as particles confined to a lattice. Their motion and interaction is governed by local rules, plus some microscopic influences, which is modeled by an independent source of noise. Such noise can either be present in nature or it represents unresolved degrees of freedom.

In general, we can distinguish two types of interacting particle systems. In the so-called *lattice models*, the local variables take only a finite number of discrete values (often called spins). The spin values are independent on the interaction with other sites. In the so-called *interacting diffusion*, the local variables take values in a continuous set, for example $\mathbb{R}$.

The model of a coupled chain of oscillators To be more specific, we consider the setting of a periodic chain of coupled bistable elements. To each site $i \in \Lambda = \mathbb{Z}/N\mathbb{Z}$, we attach a real variable $X_i \in \mathbb{R}$ describing the position of the ith particle. The configuration space is thus $\mathrm{X} = \mathbb{R}^\Lambda$.
Each particle feels a local bistable potential $V(p) = \frac{1}{4}p^4 - \frac{1}{2}p^2$. The local dynamics thus tends to push the particle towards one of the two stable positions $p = 1$ or $p = -1$. Moreover, each site is coupled to an independent noise of intensity σ. The sources of noise are described by independent Brownian Motions $B_i(t)$. The third ingredient is an interaction term which couples the particles.

Long-range interaction The novelty in our model is that the interaction between the particles is of long-range type, which means that each particle interacts with all particles which are at distance less or equal to R, where R can be very large and may go to infinity as the system size N goes to infinity.

The particle system we consider can be described as a vector-valued function $u^N = (u_1^N, u_2^N, \dots, u_N^N)$ which is a solution of the system of N coupled stochastic differential equations

$$\begin{aligned} du_i^N(t) = & \frac{\gamma}{2R^3} \sum_{j=-R}^{R} J_R(j) \left(u_{i+j}^N(t) - u_i^N(t) \right) dt \\ & - V'(u_i^N(t))dt + \sqrt{2\sigma} d\widetilde{B}_i(t), \; i \in \Lambda \end{aligned} \tag{1.7}$$

with initial condition $u^N(0) \in \mathbb{R}^N$. Here, $u_i^N(t)$ are the components of the vector $u^N(t) \in \mathbb{R}^N$, $J_R(j) \in \mathbb{R}_+$ are weights, $V(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2$ and $\widetilde{B}_i$ are independent Brownian motions. γ is a constant and $\sqrt{2\sigma}$ the intensity of the noise. The weights $J_R(j)$ describes the strength of the interaction between two particles at site i and $i + j$.

Dynamics of the nearest-neighbour model Setting $R = 1$ in our model gives the case of nearest-neighbour interactions (1.10). For fixed N and small σ, this problem has been widely studied in the literature. As this model is very related to our case of long-range interactions, we give a short overview on some results on the nearest-neighbour model. We refer to the books by Freidlin and Wentzell [55] and Olivieri and Vares [97] for details and references.

In the nearest neighbour model, we deal with a finite dimensional gradient system of the form

$$dX = -\nabla \mathcal{F}_\gamma(X) \; + \; noise \tag{1.8}$$

with potential

$$\mathcal{F}_\gamma(x) = \sum_{i \in \Lambda} V(x_i) + \frac{\gamma}{4} \sum_{i \in \Lambda} (x_{i+1} - x_i)^2 \tag{1.9}$$

This leads to the system of coupled SDEs

$$\begin{aligned} dX_i(t) \; = & \frac{\gamma}{2} \left(X_{i+1}(t) - 2X_i(t) + X_{i-1}(t) \right) dt \\ & - V'(X_i(t))dt + \sqrt{2\sigma} dB_i(t) \qquad \text{for all } i \in \Lambda. \end{aligned} \tag{1.10}$$

Due to the competition between local dynamics and coupling between different sites, a wide range of interesting behaviour was observed and investigated. First, note that presence of noise changes the behaviour of the system drastically: While without noise, we know that there exist two stable states of the system. The noise, however, causes arbitrarily small random fluctuations, which can enable transitions between stable states at large time scales. Whether such transitions are observed will depend on the timescale of interest. We call this phenomenon *metastability* and describe it in Section 1.3 of this chapter.

Second, note that, as the potential $\mathcal{F}_\gamma(x)$ in (1.9) is a polynomial of degree 4 in N variables, it can have up to 3^N stationary points. The dynamics of the system (1.10) therefore depends a lot on the coupling strength γ.

For weak coupling, the behaviour is similar to the stochastic lattice models, where one often observes spatial chaos, i.e. independent dynamics at different sites. Berglund, Fernandez and Gentz [19] have shown that the nearest neighbour model behaves like an Ising model with spin-flip dynamics for weak coupling. Also, bifurcations have been studied for the weak coupling regime, see [19] and [105] for a detailed analysis.

As the coupling strength increases, the number of equilibrium points decreases. For strong coupling (of the order N^2), the system synchronizes, in the sense that all particles assume almost the same position in their respective local potential most of the time. In this so-called synchronization regime, the coupling between the particles is so strong that there are only three relevant critical points of the potential.

Third, for large system size N, the behaviour of the nearest-neighbour interaction system is closer to the behaviour of a Ginzburg-Landau partial differential equation with noise, see [45], [106] and [112], for example. More precisely, it was shown in [66] and [67] that after suitable rescaling, the particle system (6.1) converges as N tends to infinity to a stochastic partial differential equation, the stochastic Allen-Cahn equation

$$\begin{aligned} \partial_t u(x,t) \ &= \gamma A u(x,t) - V'(u(x,t)) \ + \sqrt{2\sigma}\frac{\partial^2}{\partial_x \partial_t} W(x,t) \\ &\qquad \text{for } (x,t) \in [0,1] \times \mathbb{R}^+ \\ u(0,\cdot) \ &= u_0, \end{aligned} \tag{1.11}$$

where $A = \frac{1}{2}\Delta$ is the Laplace operator on $[0,1]$ with periodic boundary conditions, $\gamma > 0$ is the diffusion constant, V a double well potential, $\frac{\partial^2}{\partial_x \partial_t} W(x,t)$ denotes space-time white noise and $\sqrt{2\sigma}$ is the intensity of the noise.

This is the framework in which we are also working in.

The case $R = N$ We remark here shortly that the case $R = N$ corresponds to the mean-field or local mean-field models, where convergence to an SPDE cannot hold. Such models have been analysed extensively, see e.g. [40, 57, 58, 93, 83, 95, 96], and references therein.

Results presented in this work

In Chapter 5, we describe the behaviour of a large particle system with long-range interactions (1.7) in which the range of interactions R is large but fixed. Choosing the notion of weak (in the PDE sense) solutions convergence of a suitably rescaled version of (1.7) to the solution of (1.11) is obtained.

In Chapter 6, we describe the behaviour of the system (1.7) when R is also going to infinity as the system size goes to infinity. The question we address here is how fast the interaction length R can be allowed to grow in dependence on the total number of particles N and which scaling of the interaction strength is appropriate. In this setting, it is convenient to work with the notion of mild solutions. See Section 2.1.2 in the next chapter for a brief description on the two notion of solutions.
We obtain the following result (see Theorem 6.7.6, Theorem 6.7.7), which basically says that at most $\sqrt{N}$ particles can interact with each other, and in addition the interaction strength J has to satisfy a second moment condition:

Theorem 1.2.1. *Let $\mathbb{T}_h \subset [0,1]$ denote the equidistant grid on the interval $[0,1]$ with grid size $h = 1/N$ and periodic boundary conditions. Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u(x,t)$ be the solution to (1.11) and $u^h(x,t)$ the piecewise linear function obtained from the solution of the system of SDEs*

$$du^h(ih,t) = \left(\frac{\gamma}{R^3 h^2} \sum_{j=-R}^{R} J_R(j) \left(u^h((i+j)h,t) - u^h(ih,t) \right) \right) dt \\ - V'(u^h(ih,t))dt + \sqrt{\frac{2\sigma}{h}} dB_i(t), \ i \in \mathbb{T}_h, \tag{1.12}$$

with $J_R(j) = J\left(\frac{j}{R}\right)$, where J is positive and satisfies $\int J(x)x^2\, dx = 1$. If $R \sim h^{-\zeta}$ with $\zeta < \frac{1}{2}$, then
i) for all times $T > 0$, and all $p > 1$, $u^h \longrightarrow u$ in $L^p\left(\Omega, C([0,1] \times [0,T])\right)$.
ii) for all times $T > 0$, there exists an almost surely finite random

variable $\mathbb{X}$ *such that*

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \leq \mathbb{X}h^\eta$$

for $0 < \eta < \frac{1}{2} - \delta$.

1.3 Metastability

The stochastic terms in (1.7) and (1.11) reflect small scale noise that perturbs the deterministic dynamics and gives rise to a so-called *metastable behaviour*: If the system starts at one local energy minimizer, then the solution will exhibit small fluctuations around this state until it quickly crosses the energy barrier and moves into a small neighbourhood of the other minimizer. Roughly speaking, the dynamics of a system is said to be metastable if it spends a lot of time in one region (called a metastable state) before hopping to another region.

In general, the term *metastability* refers to a phenomenon observed in dynamical systems which are subject to *noise*, i.e. random perturbations, microscopic effects or unresolved degrees of freedom. Even when the noise amplitude is very small, it has a profound influence on the dynamics on the appropriate time-scale. The small noise induces *rare events*, i.e. a behaviour which is exhibited rarely with respect to the internal clock of the system. The presence of different time scales, namely the time-scale of the deterministic dynamics and the time-scale between the rare events caused by the noise, are therefore characteristic for metastable behaviour.
Examples of metastable phenomena include the condensation of supercooled vapor and the phase separation of a binary alloy, chemical reactions, conformation changes of biomolecules, bistable behaviours in genetic switches, or regime changes in climate. We will describe some of them in more detail at the end of this chapter.

Mathematical description of metastability In the mathematical description of metastability, one is interested to estimate the time when a transition from one state of the system to another is happening, how much time the transition takes and what is the optimal pathway of the transition.
First attempts to understand and model metastable behaviour mathematically go back to the works of Eyring [50] and Kramers [81], who were modeling chemical reactions. Eyring and Kramers, who were

interested in modeling reaction rates, developed a formula and first methods for calculating the expected waiting time.
There exists several methods in the mathematical description of metastability, of which we mention here the following two:
The large deviation theory of Freidlin and Wentzell [55] provides a rigorous mathematical analysis of metastability in finite dimensional (ODE) systems. Freidlin-Wentzell theory uses a pathwise approach and generalizes naturally to infinite dimensional (PDE) systems.
The other method, pioneered in Bovier, Eckhoff, Gayrard and Klein [25], is the *potential theoretic approach*, which is closely related to PDE methods. From this point of view, the transition time between two minima is simply a waiting time of one set or the exit time of the complement of this set.
To the authors knowledge, there are two classes of mathematical models where metastable behaviour was rigorously described by mathematical models up to now. The first class are lattice models with Markovian dynamics of Metropolis type, for example the Ising model with Glauber dynamics. See Olivieri and Varis [97] for details. The second class of models consists of SDEs driven by weak Gaussian white noise. Here, large deviation approaches were used, see [55] and, more recently, a potential theoretic approach [25] and [24]. This case is the background for our result in Chapter 7:

Results presented in this work

In Chapter 7 we study the behaviour of the transition times of the long-range interaction system (1.12), which we repeat here for the convenience of the reader:

$$du^h(ih,t) = \left(\frac{\gamma}{R^3 h^2} \sum_{j=-R}^{R} J_R(j) \left(u^h((i+j)h,t) - u^h(ih,t) \right) \right) dt \\ - V'(u^h(ih,t))dt + \sqrt{\frac{2\sigma}{h}} dB_i(t),\ i \in D_h, \tag{1.13}$$

Thanks to the preparatory work of Chapter 6, we can prove the convergence of transition times of our discrete system (1.12), to the transition times of the limiting SPDE (1.11).
The good thing about this result is that we know already a lot about the metastable behaviour of (1.11), indeed, there are precise estimates on the transition times of (1.11) available; they have been proved in [7] and [20] via a potential theoretic approach. Therefore, it was

convenient for use the notion of *hitting times* of metastable sets as in the potential theoretic approach for our analysis.
The main result of Chapter 7 is Theorem 7.2.4, which reads as follows:

Theorem 1.3.1. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u(x,t)$ the solution to* (1.11) *and $u^h(x,t)$ the piecewise linear function obtained from the solution of the system of SDEs* (1.12)*.*
Let $u_{\min}$ and $\tilde{u}_{\min}$ be the two minima of V. Let u_0 be close to $\tilde{u}_{\min}$. Define the continuous hitting time

$$\tau(\rho, q) := \inf_{t>0} \left\{ \|u - u_{\min}\|_{L^q([0,1])} < \rho \right\}$$

and the discrete hitting time

$$\tau^h(\rho, q) := \inf_{t>0} \left\{ \|u^h - u^h_{\min}\|_{L^q([0,1])} < \rho \right\}$$

. Then, under the conditions of Theorem 6.7.6 of Chapter 6, we have, for almost all $\rho > 0$,

$$\tau^h(\rho, q) \longrightarrow \tau(\rho, q) \qquad \text{a.s., as } h \to 0$$

and

$$\mathbb{E}\left[\tau^h(\rho, q)\right] \longrightarrow \mathbb{E}\left[\tau(\rho, q)\right] \qquad \text{as } h \to 0.$$

Examples of metastability phenomena described in the literature

The easiest example of metastability is probably the undercooling of (pure) water: As mentioned in [24], it was observed that in clear mountain lakes in the winter, the water in the lake cools well below the freezing temperature until suddenly the whole lake freezes. The stable state at negative temperature would have been ice, but the water is still found in liquid form, which is the metastable state in this case. The metastable state is very sensitive to perturbations, which can trigger an immediate transition to the stable state.
Metastability is also important in magnetization: For example, hard discs contain micromagnets which frequently change state when data is written on the disc. To estimate the rate at which magnetization changes can therefore be very important to minimize data loss. Such magnetic switching phenomena are often thermally activated and were investigated from the mathematical point of view in [80], for example.
Models containing a minimization of a multiwell potential are very frequent in chemistry. Molecules, for example, can have many configurations of their atoms, with different binding energies between the

atoms, and therefore many metastable states are observed. The corresponding multiwell potential is modeled such that each minimum corresponds to a stable configuration, and the height of the saddle between them corresponds to the necessary activation energy necessary to switch from one state to the other.

As different atomic configurations have different macroscopic properties, studying metastable transition times is of high practical importance.

A more complex situation of the same phenomenon is the folding of proteins. Here, the number of atoms is very large, and the binding structure is not easy to understand, as the protein is "folded", i.e. chains of atoms are lying very close to each other. In proteins, it is the *tertiary structure* which exhibits metastable behaviour, and each state is usually called a *conformation* of the protein.

Due to the high number of atoms, the conformational structure is often unknown, sometimes it is even unclear how many metastable states exist. The question how dynamical aspects of protein function are related to its structure is also far from being answered. In [1], the authors applied a computational method of molecular dynamics to generate a plausible all-atom model of a protein and characterized its conformations numerically. They also used a molecular dynamics simulation method to investigate the conformational variability of large proteins. Conformational variability is a problem of interest in drug design.

Metastable pathways describing the switching between two conformations have also been studied by mathematicians via numerical methods. For example, In [87], the pathways of diffusion of a carbon monoxide molecule inside a protein are described numerically. More generally, in [89] a numerical technique to compute the statical properties of the paths, now called "reactive trajectories" is presented.

Chapter 2

The L^2-gradient flow of the double well potential

The motivation for the equations we are studying is coming from physics, where thermodynamics suggests that for isothermal systems there should be a so-called "free energy" which steadily decreases and approaches a minimum at equilibrium.
As an illustrative example, let us consider the phenomenon of *phase separation*, which is the basis of the two equations we are introducing in this chapter: Given a physical system, for example a binary alloy whose temperature was lowered to a level where it can no longer exist in equilibrium in its homogeneous state. For this fixed temperature, the mixture evolves so that it eventually forms a heterogeneous mixture of two phases. Such an evolution process can be modeled by introducing a certain energy functional which is bounded from below and decreases in time.
An appropriate choice for such an energy functional is the double well potential

$$\mathcal{F}[v] = \frac{1}{2}\int_D W(v(x,t))dx \tag{2.1}$$

where the energy density $W(v)$ is smooth and non-negative with two equal minima at $v = v_1$ and $v = v_2$, corresponding to the two phases. A concise analysis shows that the two phases can be in equilibrium and in physical contact only if phase I has concentration v_1 and phase II has concentration v_2, see [54] and the references therein.
We are now looking for a law of evolution for the function $v(x,t)$ which will ensure that the energy $\mathcal{F}$ decreases. A straightforward choice is to impose that v evolve in the opposite direction of the gradient of the

functional (2.1). This evolution is generally called the *gradient flow* of $\mathcal{F}$. Below we add a very brief note on the Hilbert space setting, and refer to [5] for a detailed and concise description.

Gradient flow on a Hilbert space Let H be a Hilbert space and $\mathcal{F}$ a C^1-functional $\mathcal{F} : H \to \mathbb{R}$. We can define the Fréchet differential $\delta\mathcal{F} : H \to H'$ by

$$\mathbf{g} = \delta\mathcal{F}(v) \;\Leftrightarrow\; \lim_{w \to v} \frac{\mathcal{F}(w) - \mathcal{F}(v) - \langle \mathbf{g}, w - v\rangle}{\|w - v\|} = 0 \qquad \forall v \in H \tag{2.2}$$

(see [5], page 33). The equation

$$u'(t) = -\delta\mathcal{F}(u(t)) \qquad t > 0 \tag{2.3}$$

is called the *gradient flow* associated to the functional $\mathcal{F}$. A solution of (2.3) is a curve of steepest descent for the functional $\mathcal{F}$. It can be characterized by either of the two equivalent conditions, namely (1) the direction of the velocity u' has to be opposite to the one of the gradient or (2) the functional $\mathcal{F}$ decreases along u as much as possible. Note that formally the differential of $\mathcal{F}$ is a differential form, so an element of the dual space. The identification done in (2.3) is justified by identification of the Hilbert space with its dual.
Computing the corresponding Euler-Lagrange equation to this minimization problem, we arrive at the associated PDE to the gradient flow. The form of this PDE depends on the choice of the Hilbert space. Indeed, the same PDE can arise from two different gradient flow formulations, as we will illustrate in the following on the example of the double well potential and the Ginzburg-Landau energy:
Consider the L^2-gradient flow of the functional

$$\mathcal{F}[u] = \frac{1}{2} \int_D W(u_x(x,t))dx \tag{2.4}$$

where $D \subset \mathbb{R}$ is bounded, u_x is the gradient (in the space variable x) of a function $u : D \to \mathbb{R}$. The corresponding L^2-gradient flow reads

$$u_t = \frac{1}{2}\left(W'(u_x)\right)_x \tag{2.5}$$

which is the limit equation that we will study in Chapter 3 and 4. Inserting the regularising term $\varepsilon^2 \left(u_{xx}\right)^2$ in (2.4), we get the singular perturbation

$$u_t = -\varepsilon^2 u_{xxxx} + W'(u_x)_x \tag{2.6}$$

which is the Cahn-Hilliard equation in one space dimension. We will compare the solutions arising from (2.6) and their conjectured limit to our results in Section 2.2.3.

Alternatively, we can study the H^{-1}-gradient flow of the functional

$$\mathcal{F}[v] = \frac{1}{2}\int_D W(v(x,t))dx \tag{2.7}$$

where $D \subset \mathbb{R}$ is bounded, and $v : D \to \mathbb{R}$. The H^{-1}-gradient flow of (2.7) reads

$$v_t = \frac{1}{2}\left(W'(v)\right)_{xx}. \tag{2.8}$$

Note that (2.8) follows from (2.5) by the change of variables $v = u_x$. As the functional $\mathcal{F}$ is not convex, the PDEs (2.5) and (2.8) are not well-posed.
Inserting the regularising term $|u_x(x)|^2$ in (2.7) gives the Ginzburg-Landau energy (see (2.13) below for details) whose H^{-1}-flow is the Cahn-Hilliard equation

$$v_t = \Delta(-\Delta v + W'(v)) \tag{2.9}$$

and whose L^2-gradient flow is the Allen-Cahn equation

$$v_t = \Delta v - W'(v), \tag{2.10}$$

whose stochastic perturbation is the object of study in Chapter 6.
To complete the confusion of names, we will sometimes call (2.8) an "unperturbed version of the Cahn-Hilliard equation", see Section 2.1.3 for details.

Inserting a regularising term in the energy not only has a positive effect on the existence of solutions to the associated PDE, but it also posesses the physical meaning of penalizing too rough transitions between phases. For reasons of clarity we have avoided adressing the role of the small coefficient ε in front of the regularising term in the above discussion, which is of course important to consider in applications: In phase transitions, ε denotes the width of the transition layer, in image processing, adjusting the size of ε is important to make sure that the algorithm detects the correct boundaries of objects in the image, see Section 2.3 for details.

Moreover, the choice of coefficients has also an influence on the mathematical properties of the functional: As shown by Modica and Mortola in [91] and [90], the energy

$$\mathcal{E}_\varepsilon(u) = \int_D \frac{\varepsilon}{2}\left|\nabla u(x)\right|^2 + \frac{1}{\varepsilon}W(u(x))\,dx \tag{2.11}$$

Γ-converges to the perimeter functional

$$\mathcal{E}(u) := \begin{cases} Per(\Sigma) & \text{if } u(x) = \mathbf{1}_\Sigma(x) \text{ for some } \Sigma \subset D \\ +\infty & \text{otherwise.} \end{cases} \tag{2.12}$$

Γ- convergence is an important tool in the Calculus of Variations. We mention here only that Γ- convergence of a sequence of functionals implies that the sequence of minimizers of the approximating functionals converges to the minimizer of the limit functional.

2.1 The Allen-Cahn and the Cahn-Hilliard equation

In this section, we state some basics facts about the Allen-Cahn equation (2.10) and the Cahn-Hilliard equation (2.9), which we will refer to in Chapters 3 - 6 of this work.
As stated above, both equations are gradient flows of the same energy functional, namely

$$\mathcal{E}(u) = \int_D \frac{1}{2} |\nabla u(x)|^2 + W(u(x))\, dx. \tag{2.13}$$

which is often called *Ginzburg-Landau energy.* Here, W denotes the double well potential as before. Candidates for minimizers of (2.13) prefer to be constant in space (to keep gradients small) and prefer to take values in the two minima ± 1 of W (to minimize the bulk energy). The functional (2.13) is widely used in physics literature, for example in problems of surface tension, wetting, nucleation, spinodal decomposition, and liquid-liquid interfaces. One of the first appearances is believed to be in an 1893 article of van der Waals (see [107] for a commented translation). It is said that he pioneered a thermodynamical treatment of non-uniform systems.
Below, we describe briefly the specific properties and applications of the Allen-Cahn equation and the Cahn-Hilliard equation. An important difference between them concerns the total mass $\int u(x,t)\, dx$, which can mean also the total fraction of atoms within a given structure or the total magnetization of a material, depending on the application: The Cahn-Hilliard model conserves the total mass, the Allen-Cahn model doesn't. An advantage of the Allen-Cahn equation as a second-order equation is the availability of comparison principles, which is not the case for the fourth-order Cahn-Hilliard equation.

2.1.1 The Allen-Cahn equation

The Allen-Cahn equation is a model for phase separation and the evolution of interfaces between the phases for systems without mass conservation. The equation was introduced in [2] to model the growth of grains in crystalline materials near their melting points.
We model such systems by a function u which describes the system, and by an energy associated to each configuration u. Classically, a refinement of the energy (2.13) is used, namely (2.11), which we restate here for the convenience of the reader:

$$\mathcal{E}(u) = \int_D \frac{\varepsilon}{2} |\nabla u(x)|^2 + \frac{1}{\varepsilon} W(u(x))\, dx. \tag{2.14}$$

Applying periodic or Neumann boundary conditions, the energy (2.14) has two minima, $u = \pm 1$, which we call *phases*, and the parameter ε corresponds to the width of the interfaces between the different phases. Often, the term

$$\int_D \frac{1}{\varepsilon} W(u(x))\, dx. \tag{2.15}$$

is called the *potential energy* and the term

$$\int_D \frac{\varepsilon}{2} |\nabla u(x)|^2 \, dx. \tag{2.16}$$

is called the *kinetic energy*. The kinetic term (2.16) penalizes transitions between the phases.
By enforcing $\lim_{x\to-\infty} u(x) = -1$ and $\lim_{x\to\infty} u(x) = 1$, the solution u has to change phase at least once. The minimizer of (2.14) is then given by functions of the form

$$m_a(x) \; = \; \tanh\left(\frac{x-a}{\sqrt{2\varepsilon}}\right). \tag{2.17}$$

In case that we have n such transitions, generally all of them will have a width of order ε around some points $a_1, \dots, a_n$. These points will appear as "jump points" in the setting of Chapter 3, where $\varepsilon = 0$.
The deterministic Allen-Cahn equation on a bounded interval $D \subset \mathbb{R}$ is the L^2-gradient flow of the energy (2.13), that, for $\varepsilon = 1$, reads as

$$\begin{aligned} \partial_t u(x,t) &= \partial_x^2 u(x,t) - u^3(x,t) + u(x,t) \qquad (x,t) \in D \times (0,T] \\ u(.,0) &= u_0 \qquad x \in D \end{aligned} \tag{2.18}$$

plus suitable boundary conditions.

The dynamics of solutions to this equation is as follows: solutions will very quickly be drawn to a configuration that is almost everywhere close to ± 1 with transitions of width of order ε around some points a_i. Note that these points a_i move very slow. When two such a_i meet, the phase boundaries annihilate and the configuration looses energy. This phenomena is called *coarsening*. For a detailed study, see for example Carr and Pego [34] and references therein.
On the level of the gradient flow, we can also explain coarsening like this: We see that for small values of ε, the energy of a configuration is given by the surface of the separation layer. Therefore, the dynamics tends to decrease the surface, and this is mathematically encoded in the Γ-convergence of the energy functional (2.11) to the perimeter functional (2.12). This also explains on an intuitive level the relationship to the mean curvature flow, a topic which we will, however, not address in this work.

2.1.2 The stochastic Allen-Cahn equation

Adding a small space-time white noise to the right hand side of (2.18), we get the *stochastic Allen-Cahn equation*

$$\partial_t u(x,t) = \partial_x^2 u(x,t) - u^3(x,t) + u(x,t) + \sqrt{2\sigma}\frac{\partial^2}{\partial_x \partial_t} W(x,t) \quad (2.19)$$

The stochastic Allen-Cahn equation is used in models of transition of phases and the evolution of interfaces between the phases, see for example Brassesco and Buttà [27]. Other popular names for (2.19) in the literature are stochastic Ginzburg-Landau equation, Φ^4 model or stochastic quantization equation. The name stochastic Ginzburg-Landau equation is often used to a version derived from an energy like (2.13), where the coefficient ϵ in front of the diffusion and a $\frac{1}{\epsilon}$ in front of the double well term is reall small and not set to one. The name Φ^4 model is frequently used in literature which deal with applications in quantum field theory, see for example Faris and Jona-Lasinio [51] and Cassandro, Olivieri and Picco [35] or also [69].
The stochastic Allen-Cahn equation has been treated a lot in physics literature, see for example [75]. Rigorous results in one space dimension have been obtained by Funaki [56], who interprets (2.19) as the motion of a string in a stochastic environment or by Brassesco, De Masi and Presutti [28]. In [117], the invariant measure of (2.19) is studied. The author points out that the invariant measure can be given very explicitly thanks to the gradient structure of the Allen-Cahn equation. He shows that the invariant measure is absolutely

continuous with respect to the distribution of a Brownian Bridge and the Radon-Nikodym density corresponds to the potential energy of the path.

Existence of solutions Two notions of solutions, namely *weak solutions* and *mild solutions*, are most often used when studying the stochastic Allen-Cahn equation. We refer to Definition 5.1.1 in Chapter 5 for a definition of weak solutions and to Definition 6.5.1 in Chapter 6 for a definition of mild solutions. Weak solutions are often called *weak (in the PDE sense) solutions* or *variational solutions*: they are used when functional analytical arguments come into play. Furthermore, weak solutions are convenient when Galerkin approximation methods, such as the Finite Element method in numerical analysis of PDEs, are used, see [84]. Bensoussan and Temam [18] and Pardoux [101] proved the existence of weak (in the PDE sense) solutions of the stochastic Allen-Cahn equation in one space dimension using the theory of monotone operators. We use their ansatz in Chapter 5.

Mild solutions are also called *integral solutions*, they are useful when a lot is known about the semigroup of the (diffusion) operator appearing in the SPDE. The existence, uniqueness, and regularity of mild solutions of the one-dimensional stochastic Allen-Cahn equation was proved in Gyöngy and Pardoux [68]. This ansatz is followed by us in Chapter 6. Approximations of such parabolic SPDEs were studied in one space dimension by Funaki [56] and Gyöngy [66], and in Gyöngy and Millet [67] for equations driven by a d-dimensional Brownian Motion with $d > 1$.

In more than one dimension, the stochastic Allen-Cahn equation (2.19) does not have a solution in the classical sense, and one needs to use renormalisation in order to obtain non-trivial results. In two dimensional case this was done by Da Prato and Debussche [38] by using the Wick product in the non-linear term. In dimension three a notion of solution was provided by the framework of Regularity Structures [69], which allows to treat a large class of non-linear SPDEs. An alternative approach to the equation was provided by [36] using paracontrolled distributions (see [65] as an introduction).

In the recent work [70], Hairer and Matetski developed a systematic approach of spatial discretisations of non-linear SPDEs whose solutions are provided by regularity structures. As an application they proved convergence of the dynamical Φ^4_3 model to the continuous one. The latter result was obtained independently by Zhu and Zhu [120] using paracontrolled calculus.

The stochastic Allen-Cahn equation appears in Chapter 6 and 7 of this work.

2.1.3 The Cahn-Hilliard equation

The Cahn-Hilliard equation was developed to model phase separation and subsequent phase coarsening of binary alloys [32, 33], see also [94] and references therein. It reads

$$\begin{aligned} v_t &= \Delta(-\varepsilon^2 \Delta v + W'(v)) && \text{in } D \times (0,T) \\ \partial_\nu v &= \partial_\nu(-\varepsilon^2 \Delta v + W'(v)) = 0 && \text{in } \partial D \times (0,T) \\ v_0 &= v(0) && \text{at } D \times \{0\} \end{aligned} \tag{2.20}$$

where W is a double well potential and ∂_ν denotes differentiation normal to ∂D.
In the physical model, the solution $v(x,t)$ represents the concentration of one of the two metallic components of the alloy, ε is an "interaction length". The boundary conditions reflect two things, namely (1) the physical fact that the mixture cannot penetrate the walls of the container and (2) the thermodynamical requirement that the total "free energy" of the mixture decreases in time (when there is no interaction between the alloy and the wall of the container).
The boundary conditions also imply that the total mass is conserved, i.e.

$$\int_D v(x,t)dx = const \tag{2.21}$$

(2.20) is often derived as constrained H^{-1}-gradient flow of the Ginzburg-Landau free energy (2.13), the constraint being (2.21)
For the classical case of $W(v) = \frac{1}{8}(1-v^2)^2$, one finds in the literature the name "spinodal interval" for the backward-parabolic region, and $\left(-1, -\frac{1}{\sqrt{3}}\right) \cup \left(\frac{1}{\sqrt{3}}, 1\right)$ is often referred to as "metastable intervals".

Note that every constant function is a stationary solution of (2.20), which is asymptotically stable if the initial datum is contained in the metastable intervals. If it is contained in the spinodal interval, the stationary state is unstable, see Fife [54] and Grant [64] for details and references. To mention one result in this direction, it has been shown by Sander and Wanner [108] that, if a solution of the Cahn-Hilliard equation starts sufficiently close to such a stationary solution in the spinodal interval, then it will "almost certainly follow the corresponding solution of the linearized equation up to an unexpectedly large distance from that stationary state". We refer to [108][p. 2186] for a detailed statement, which explains what is meant by "almost certainly" and "unexpectedly large distance".

Dynamics of solutions The dynamics of the Cahn-Hilliard equation has been studied by many authors, especially aspects like phase separation, which is often called *spinodal decomposition* in this context, and the subsequent slow interface dynamics. Generally speaking, after the formation of a fine mixture of phases, regions in the system that have the same phase are merging, and so, the transition zones between different phases are reduced. This coarsening effect is observed in different systems in physics, like phase separation of metallic alloys, liquids, or polymer mixtures. Coarsening happens on a larger time scale than the initial separation of phases, for an overview see, for example, [10] and references therein. Experimental and numerical measurements suggest that, typically, coarsening rates behave according to power laws. Rigorous growth laws has been established in the form of a weak upper bound by Kohn and Otto [79] for the Cahn-Hilliard equation in the case of equal volume fractions.

As an example in a setting close to ours, we give the size of those different time scales explicitly for the dynamics of a Cahn-Hilliard equation in one space dimension with small ε, which has been studied numerically by [14]: First, for time intervals of order $O(\varepsilon^2)$ we observe the formation of microstructures in the spinodal interval which drastically reduce the initial energy without too large deviation of $u^\varepsilon(t)$ from the initial data in the L^∞-norm. At a larger time scale (time intervals of order $O(1)$), the evolution appears to be quite slow and corresponds to the solution of the heat equation, resulting in a reduction of the energy of the system also in the unwrinkled region. Last, the evolution at a later stage, when the number of interfaces is reduced, becomes very slow, the authors give no specific time scale but refer to other works, where the motion of interfaces was reported to be at a speed of order $O(e^{-c/\varepsilon})$ for some constant c, so exponentially small in the interface length parameter ε.

2.2 Forward-backward parabolic equations

We now return to the original object of study, which is a certain forward-backward parabolic PDE arising as the L^2-gradient flow of a non-convex potential. The name *forward-backward parabolic equations* is used as heuristically, the equation behaves like a time-reversed heat equation in the part where convexity of the potential fails.
Forward-backward parabolic equations are typically written in the form of (2.8), i.e.

$$v_t = \frac{1}{2}\left(\Phi'(v)\right)_{xx} \quad \text{in} \quad D \times [0,T] \tag{2.22}$$

for general non-convex potentials Φ. They arise in different mathematical models in phase transition [30], population dynamics [99], [98], oceanography [6] or image processing [47].

The majority of the literature deals with two types of non-convex potentials $\Phi(p)$, namely potentials which are convex for small values of p and become concave when a critical value is exceeded, and double well potentials, which are concave for small values and convex for large values of p. In this work we consider double well potentials, but we give a brief overview on the results for the other type using the famous example of the Perona-Malik equation, in Section 2.3.

As mentioned already, the lack of forward parabolicity in (2.22) gives rise to ill-posed problems, when one expects the development of singularities or the formation of microstructures.
The are a few existence results for initial data which takes values only in the convex part of the potential, for example [76] in the case of Perona-Malik type potentials. Our result in [60], which we present in Chapter 3 of this work, establishes a similar existence result, but for double well potentials.
The question of uniqueness was examined, for example in [71], who showed the existence of infinitely many weak L^2-solutions for some piecewise linear potential, or several results in for Perona-Malik type potentials, see Section 2.3. Several authors study Young measure solutions, see, for example, [92, 44, 109].

Usually, well-posedness of forward-backward parabolic equations is achieved by regularisation, see [114] for a general overview and some specific regularisations related to our work below. Other approaches involve convexification procedures, see [72] and [44] for example, or by defining appropriate notions of weak solutions as for example in [16].
Regarding the existence of solutions to the forward-backward parabolic PDEs (2.5) and (2.8), which are derived from the L^2- or H^{-1}-gradient flow of double well potential, several regularisations have been studied in the literature. Note that, obviously, different regularisation procedures don't necessarily lead to the same solution, not do the solutions to the regularised problems have the same properties. In the following, we will focus on three types of regularisations and compare them to the results we obtain in Chapter 3.

2.2.1 Viscous approximations

The idea here is to study regularised problems with regularisation term $-\varepsilon\Delta u_t$, which is sometimes called the *viscous problem* associ-

ated with the original forward-backward problem, by analogy with the theory of conservation laws. Its limit is then called the *vanishing viscosity limit*.
This type of regularisation has been studied by Plotnikov [104] and Evans and Portilheiro [49], and later by [110, 111] and [113], see also the references therein.
We will describe the results by Plotnikov [104] and Evans and Portilheiro [49] now more closely, as we will see later on that our limit problem (see Theorem 3.3.5 and equation (3.38) in Chapter 3) coincide with the limit problem obtained by these authors in the one-dimensional case.
Consider the boundary value problem

$$\begin{aligned} v_t^\varepsilon &= \Delta W'\left(v^\varepsilon\right) + \varepsilon \Delta v_t^\varepsilon && \text{in } D \times [0,T] \\ 0 &= \partial_\nu \left(W'\left(v^\varepsilon\right) + \varepsilon v_t^\varepsilon\right) && \text{on } \partial D \times [0,T] \\ v^\varepsilon &= v_0^\varepsilon && \text{on } D \times \{0\} \end{aligned} \tag{2.23}$$

where $D \subset \mathbb{R}^n$ is a bounded regular domain.
In [104] it was shown that (2.23) has a family of solutions $v^\varepsilon(x,t)$ uniformly bounded in L^∞. Moreover, as $\varepsilon \to 0^+$, the sequence $v^\varepsilon(x,t)$ converges to a measure-valued solution of (1.2). One then obtains an entropy formulation for the solutions of the original forward-backward problem.
The limit points of the set of solutions to (2.23) can be characterized with the aid of the variable $w = W'(v)$, which can be interpreted as the temperature distribution in the medium: divide the graph of the inverse of W', in three branches $S_i(w)$ chosen in a way that $S_1(w)$ and $S_2(w)$ are monotonically increasing and $S_0(w)$ is monotonically decreasing. Then the following statement holds:

Theorem 2.2.1 (Plotnikov)**.** *There exists three measurable functions $\lambda_0, \lambda_1, \lambda_2$ such that for a. e. point $(x,t) \in D \times (0,T]$ we have*

(i) $0 \leq \lambda_i \leq 1$

(ii) $\sum_{i=0}^{2} \lambda_i = 1$

(iii) Passing as necessary to a further subseqence,

$$F(v_{\varepsilon j}) \rightharpoonup \bar{F} := \sum_{i=0}^{2} \lambda_i F\left(S_i(w)\right) \quad \text{weakly} * \text{ in } L^\infty\left(D \times [0,T]\right) \tag{2.24}$$

for each continuous function F.

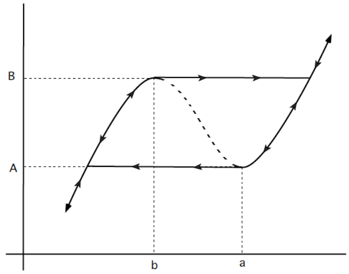

Figure 2.1: *Example of a hysteresis loop, taken from [49][Figure 4]*

(iv) Defining $w^{\varepsilon j} := W'(v^{\varepsilon j}) + \varepsilon v_t^\varepsilon$, *one has*

$$w^{\varepsilon j} \to w \quad \text{strongly in } L^2\left(D \times (0,T)\right). \tag{2.25}$$

Moreover, w is related to v in the following way

$$\begin{aligned} v_t &= \Delta w && \text{in } D \times [0,T] \\ \partial_\nu w &= 0 && \text{on } \partial D \times [0,T] \\ v(x,0) &= v_0 && \text{on } D \times \{0\}. \end{aligned} \tag{2.26}$$

In other words, passing to the limit as $\varepsilon \to 0$ yields atomic Young measure solutions to $v_t = \Delta W'(v)$. Plotnikov's limit problem also explains the hysteresis phenomenon: The limit function v is a generalized solution to the differential equation $v_t = \Delta w$, $w \in [W'^-(v), W'^+(v)]$ where $W'^-(v)$ is the greatest minorant and $W'^+(v)$ is the least majorant of $W'(v)$ in the class of nondecreasing functions. The domain enclosed by the graphs of $W'^{\pm}(u)$ on the (v,w) plane is referred to as the *hysteresis loop*. It is a curvilinear quadrilateral bounded above and below by segments of the straight lines $w = \{W'(A), W'(B)\}$ where A rsp. B denote the values of the local minimum rsp. local maximum of W'. The lateral sides of the hysteresis loop are the arcs $v = S_1(w)$ and $v = S_2(w)$. We display the illustration given in [49] in Figure 2.1 for the convenience of the reader.

Comparison of our result to the viscous approximation

Let us now consider an illustrative example of two pure phase regions studied in [49]. Choose two open regions V_1, V_2 in $D \times [0,T]$ with a smooth n-dimensional interface $\Gamma := \overline{V_1} \cap \overline{V_2}$ Assume that

$$\begin{aligned} \lambda_0 &= 0 \qquad \text{a.e. in } D \times [0,T] \\ \lambda_1 &= 1 \qquad \text{a.e. in } V_1 \\ \lambda_2 &= 1 \qquad \text{a.e. in } V_2 \end{aligned}$$

In terms of Theorem 2.2.1, we are assuming:

$$\begin{aligned} v^{\varepsilon j} &\longrightarrow S_1(w) \quad \text{a.e. in} \quad V_1 \\ v^{\varepsilon j} &\longrightarrow S_2(w) \quad \text{a.e. in} \quad V_2 \end{aligned}$$

and the two pure phase regions V_1 and V_2 are separated by a smooth free boundary Γ.
Then we get the following behaviour of the free boundary:

Theorem 2.2.2 (Evans & Portilheiro). *Let $\overline{\nu} = (\nu^1, \ldots, \nu^n, \nu^{n+1}) = (\nu, \nu^{n+1})$ denote the unit normal in $\mathbb{R}^{n+1}$ pointing into V_1. Let v , $w = W'(v)$ be as in (2.26), in particular let v_1 rsp. w_1 denote the values of v rsp. w along the Interface Γ from within V_1, and v_2 rsp. w_2 denote the values along the Interface Γ from within V_2.*
Then we have

$$\begin{aligned} S_1(w)_t &= \Delta W'(v) \quad \text{in} \quad V_1 \\ S_2(w)_t &= \Delta W'(v) \quad \text{in} \quad V_2 \end{aligned} \tag{2.27}$$

Furthermore,

$$W'(v_1) = W'(v_2) \quad \text{along } \Gamma \tag{2.28}$$

and we write $W'(v) = W'(v_1) = W'(v_2)$ along Γ.
Let $[v] := v_1 - v_2$ rsp. $[\nabla W'(v)] := \nabla W'(v_1) - \nabla W'(v_2)$ denote the jump of v rsp. $\nabla W'(v)$ across the interface. Then

$$\nu^{n+1} = \nu \cdot [\nabla W'(v)] \quad \text{along } \Gamma \tag{2.29}$$

and

$$\begin{aligned} \nu^{n+1} &= 0 \quad \text{if } v \neq A, B \\ \nu^{n+1} &\geq 0 \quad \text{if } v = A \\ \nu^{n+1} &\leq 0 \quad \text{if } v = B \end{aligned} \tag{2.30}$$

where A rsp. B denote the value of the local minimum rsp. local maximum of W', i.e. B is the endpoint of the curve S_1 and A is the starting point of the curve S_2.

Note that the above example illustrates Theorem 2.2.1 in the case when the backward-parabolic region has codimension 1. In the one-dimensional case, this means that there exist at most countably many jump points.

Comparison of the one-dimensional case In the following, we will compare the limit equations (2.27) and (2.28) to our limit equation for $v = u_x$ (equation (3.38)) in the one-dimensional case.
In the case that there exists exactly one jump a, the limit problem (3.38) we obtain as a consequence of Theorem 3.3.5 in Chapter 3 reads

$$\begin{aligned} (i) \qquad v_t &= W'(v)_{xx} && \text{in } D \setminus \{a\} \\ (ii) \quad W'(v(a^-)) &= W'(v(a^+)) && \text{at } \{a\} \\ (iii) \quad W'(v(a^-))_x &= W'(v(a^+))_x && \text{at } \{a\}. \end{aligned}$$

We observe that (i) is in the simple phase case described above obviously equal to (2.27). Equation (2.28) says nothing else than $W'(v(a^-)) = W'(v(a^+))$ at the jump point a of v, which is (ii). Finally, for $W'(v) < A$ rsp $W'(v) > B$ we have with (2.29)

$$0 \overset{(2.30)}{=} \nu^{n+1} = \nu \cdot \big(W'(v(a^-))_x \; - \; W'(v(a^+))_x\big) \quad \text{at } \{a\}$$

which gives (iii).
Furthermore, note that in the 1D case equation (2.29) and (2.30) reduce to

$$[v_t] = 0 \quad \text{on } \Gamma.$$

This says in particular that the interface, i.e. the jump points of v do not move as time passes. In the language of Section 3.2.2 in Chapter 3, this is given by the stability of solutions with initial conditions in the region $\left(\frac{1}{\sqrt{3}}, \frac{2}{\sqrt{3}}\right)$.

2.2.2 Time discretisation

The following method is based on the so-called *minimizing movements*, a notion of energy-based motion which generalizes an implicit-time scheme for the approximation of solutions of gradient flows. The terminology *minimizing movement* has been introduced by De Giorgi in a series of papers devoted to mathematical conjectures [42]. One can think about this approach as a generalisation of an implicit Euler discretisation in the time variable. See [4] for a detailed exposition on the minimizing movement method.
In [78], Kinderlehrer and Pedregal used this method for proving the existence of solutions to evolution problems, and their approach was later generalized by Demoulini [44] to forward-backward parabolic equations of the type (2.5) in several space dimensions. Here, the evolution problem is approximated by a sequence of stationary problems, the solutions of which are in turn interpreted as minimizers of

variational principles. More precisely, the time-discretized version of (2.5) is the Euler equation of a non-convex variational principle which at each times step in minimized. The minimizer solves the stationary problem and approximates the solution of (2.5) within time h. By taking arbitrarily small time steps, one passes from the stationary problem to the evolution problem.
In [44] it was shown that - under quadratic growth conditions for the nonlinear term - a Young measure solution obtained by this approach has support in the set where the potential coincides with its convex hull.
The solution $u = \lim_{\tau \to 0} u^\tau$ is therefore also a solution to the gradient flow of

$$\mathcal{F}[u] \;=\; \int_D W^{**}(\nabla u)\, dx. \tag{2.31}$$

where W^{**} is the convexification of the potential Φ, i.e.

$$W^{**} = \max \{ f \leq \Phi : f \text{ is convex } \}. \tag{2.32}$$

We conclude that the the minimizing movement method selects a global minimum, and as $\mathcal{F}$ vanishes on all functions whose gradients assume only values ± 1, this implies that all 1-lipschitz functions do not move.

Comparison with our results Taking into consideration the simulations in [14], we expect that in our model, differently to the solutions obtained by time discretisation, solutions with $|(u^h_x)_i| < \frac{1}{\sqrt{3}}$ will move not much in time, and microstructures of slope near ± 1 will develop as the energy decreases quickly until being quasi 0.
As we expect a non-stationary solution at least in the case of an initial datum with values in the convex zone of the functional, the solutions given by minimizing movements cannot display these phenomena.

2.2.3 Regularisation with higher order terms

In this section, we look at a special regularisation of the forward-backward parabolic equation (2.8) by inserting the higher order term $\varepsilon^2 u_{xxxx}$, which leads to the fourth order PDE

$$u_t = -\varepsilon^2 u_{xxxx} + W'(u_x)_x \tag{2.33}$$

We note that setting $u_x = v$ (this is a one-dimensional equation), equation (2.33) becomes the Cahn-Hilliard equation by differentiation).
Equation (2.33) is the L^2-gradient flow of the energy

$$\mathcal{F}_\varepsilon[u] := \frac{1}{2}\int_0^1 \varepsilon^2 (u_{xx})^2 + W(u_x)\,dx. \tag{2.34}$$

which is strictly convex in u_{xx}.
In [41], De Giorgi conjectured the existence of a pointwise limit for solutions $u^\varepsilon(x,t)$ of (2.33), which makes it interesting to look at the dynamics of this equation for small ε or $\varepsilon \to 0^+$.
Some works in this direction are [109, 14, 12]. In [109], the asymptotic behavior of measure valued solutions is studied and a general trend to equilibrium for a weak solution to the unperturbed equation in the sense of Young measures is proved. In [12] the authors are able to pass to the limit in (2.33) as $\varepsilon \to 0^+$, under the assumption that the initial data (which now depend also on ϵ) converge to $\overline{u}$ in a suitable energetic sense.
In [14], a limit equation for solutions $u^\varepsilon(x,t)$ is conjectured. We will now shortly describe this limit equation for the case of the double well potential. We choose $D = [0,1] \subset \mathbb{R}$ and compare it with our limit equation (3.34). The reader is adressed to [14] for the precise statements.

First, we need to specify a class $\mathcal{A}$ of initial data, such that we can associate a unique local solution to (2.5) to each initial datum of this class. The basic idea behind this definition is that for any $u \in \mathcal{A}$ the limit equation is forward-parabolic away from the singular points.

Definition 2.2.3. [14, Def 5.1] We write $u \in \mathcal{A}$ if $u \in Lip(D)$ and there exists a natural number $M > 0$ and a partition $a_0 = 0 < a_1 < \ldots < a_{M-1} < a_M < a_{M+1} = 1$ such that $u \in W^{2,2}(a_{j-1}, a_j)$ for all $j = 1 \ldots M+1$ and either

$$\sup u_x < -\frac{1}{\sqrt{3}} \text{ in } (a_{j-1}, a_j) \quad \text{and} \quad \inf u_x > \frac{1}{\sqrt{3}} \text{ in } (a_j, a_{j+1})$$

or

$$\inf u_x > \frac{1}{\sqrt{3}} \text{ in } (a_{j-1}, a_j) \quad \text{and} \quad \sup u_x < -\frac{1}{\sqrt{3}} \text{ in } (a_j, a_{j+1}).$$

We call $a_1, \dots a_M$ singular points of u or jump points of u_x. Note that

$$\inf_{x \in D \setminus \{a_1, \dots, a_M\}} W''(u_{0x}(x)) \;>\; 0.$$

From the simulations performed in [14] with very small values of ε, the limit of (2.33) as ε goes to zero is conjectured to be the following:

Proposition 2.2.4. *[14, Prop 5.4] Consider an initial data $\nu \in \mathcal{A}$ with singular points $a_1(0) = a_1^0, \dots, a_M^0$. Define the jump set of u_x as $J_{u_x}(t) := \{a_1(t), \dots, a_M(t)\}$. Then the L^2 gradient system associated with $\mathcal{F} = \frac{1}{2}\int_D W(u_x)\,dx$ in the set $\mathcal{A}$ reads as follows*

$$\begin{aligned}
u_t &= \frac{1}{2} W'(u_x)_x && \text{in } D \setminus J_{u_x}(t) \times [0,T] \\
u_x(a_j^{t,-}) \; &\neq u_x(a_j^{t,+}) && \text{at } j \in \{1 \dots M\} \\
u_x(a_j^{t,\pm}) &\in \{-1, 1\} && \text{at } j \in \{1 \dots M\} \\
u_x(0^+, t), u_x(1^-, t) &\in \{p \in \mathbb{R} : W'(p) = 0\} && \\
u(\cdot\,, 0) = u_0(\cdot\,)\,,\ a_j(0) &= a_j^0 && \text{in } D \times \{0\}.
\end{aligned} \tag{2.35}$$

Furthermore, solutions to (2.35) preserve the average, i.e.

$$\frac{d}{dt}\int_D u(x,t)dx \;=\; 0.$$

Comparison to our limit equation

Equation (2.35) is different from our limit problem (3.34): In particular, the third line of the above equation imposes the interior boundary condition

$$W'(u_x(a_j^-)) \;=\; W'(u_x(a_j^+)) \;=\; 0 \tag{2.36}$$

which is a stronger condition than our $W'(u_x(a_j^-) \;=\; W'(u_x(a_j^+))$. Due to these addition interior boundary conditions, it is in general not possible to require that the jump points a_j do not move in time, a property that followed in our case from Proposition 3.2.10 (the stability estimate). In fact, in the third time scale of the dynamics of the limit equation, in difference to [14], we expect no coarsening of our system as our jumps do not move in time.

It should be remarked that in Definition 2.2.3, Bellettini et al required only that the solution assumes no values in the spinodal interval, so in particular no upper bound on gradients like in (3.34) needs to be assumed.
However, the conjectured limit equation obtained in [14] was tested numerically only on initial data with sufficiently small gradients, which does not give us information to compare their conjectures with our discussion in Section 3.2.2 of Chapter 3, where we motivated that an upper bound on the initial data is necessary.

2.3 Examples of forward-backward parabolic equations

In the following, we give both an example where forward-backward parabolic PDEs appear naturally in real-world applications, namely the Perona-Malik equation studied in image processing and population dynamics, and an example where it is not feasible to work with the forward-backward parabolic PDE itself, but a regularisation (here: the Cahn-Hilliard equation) has to be chosen.

The Perona-Malik equation

The Perona-Malik equation is the formal L^2-flow of the potential

$$\Phi(p) = \log\left(1 + p^2\right) \tag{2.37}$$

for which the non-convex region is $(-\infty, -1) \cap (1, \infty)$. Note that in the case of the Perona-Malik potential, the convex region is that with values of p, while in the case of the double-well potential, the region with small p was backward-parabolic.
The L^2-gradient flow of (2.37) in one space dimension reads

$$\begin{aligned} u_t - \left(\rho(u_x^2)u_x\right)_x &= 0 \\ u(x,0) = I(x) & \end{aligned} \tag{2.38}$$

with

$$\rho = \frac{1}{1 + |\nabla u|^2} \tag{2.39}$$

as a classical choice.
The initial condition $I(x)$ is interpreted as the grayscale intensity of the given image. The equation is named after Perona and Malik [103], who first used it for *denoising*, i.e. the improvement of a digital image by removing short-scale oscillations (which are called “noise”).
Reformulating the gradient flow formulation into the following PDE on the space-time cylinder $D \times [0, T]$,

$$u_t - c(u_x^2)u_{xx} = 0 \tag{2.40}$$

with $c(p) = \rho(p) + 2u\rho'(p)$. We can easily observe that the equation (2.38) is forward parabolic at a point $x \in D$ if $c(u_x^2) > 0$ and backward parabolic if $c(u_x^2) < 0$. Note that there is a threshold value K such that $c(u_x^2) > 0$ for $u_x^2 < K^2$ and $c(u_x^2) < 0$ for $u_x^2 > K^2$.

Translated in the language of images, this means that the Perona-Malik equation is smoothening edges if they correspond to small gradients, and "sharpening" or "enhancing" edges when they correspond to regions of large gradients. This seems very natural in application to image segmentation: While the noise is smoothened, the boundaries of objects are enhanced and the objects in the image can be clearly detected.

However, (2.38) is an ill-posed equation, and the edge enhancement via a time-reversed heat-type equation it is not justified on rigorous level. There are several non-existence results (see [77] for an overview) for weak solutions even in cases when numerical experiments were successful. Indeed, the original scheme of Perona and Malik did not exhibit significant instabilities; in fact, when computed over large enough time, the solution appears to tend to a piecewise constant solution, representing a simplified image with sharp boundaries - the very goal of such an algorithm. Looking more carefully, there were simulations which showed slight oscillations in first derivatives, but none of the large-scale oscillations, which are usually associated with ill-posedness, appear.

To sum up the above discussion, it seems that there exist stable numerical schemes for initial-value problems which don't have a weak solution. This is often called the *Perona-Malik Paradox*.

There were several attempts to explain this paradox, for example by introducing a time-delay regularisation, see [3, 11]. In this case, the term u_x^2 is replaced by an average of its past values. The numerical results seem to depend on the type of mesh refinement used.

Nevertheless it seemed unreasonable to abandon the Perona-Malik model completely, as it appeared not only in image processing but also in other fields, and also the H^{-1} flow of the (2.37) arises independently in a model of aggregating populations in population dynamics [98], see also [110, 111].

Lots of effort was made to study the limiting behaviour as the grid size h goes to zero, the existence theory of the continuous equations and appropriate regularisations, see also [17, 13, 14, 61, 63, 76].

Let us mention in particular [47], where a meaningful continuum limit to the semidiscrete scheme proposed by Perona and Malik is proposed. This limit can be interpreted as the gradient flow of an energy, and it involves a system of heat equations which are coupled to each other through nonlinear boundary conditions.

Cahn-Hilliard inpainting

Image inpainting is the process of filling in missing parts of damaged images base on information from the surrounding areas. This can be achieved, for example, by connecting contours of constant grayscale image intensity (called isophotes) to each other across the inpainting region, so that gray levels at the edge of the damaged region get extended to the interior continuously.
The work of Bertalmio, Sapiro, Caselles and Ballester [21] introduced a model based on nonlinear PDEs, which was designed to imitate the techniques of museum artists who specialize in restauration of images. Here, the direction of isophotes is imposed as a boundary condition at the edge of the inpainting domain.
In [22, 23], a Cahn-Hilliard inpainting model was introduced, which had the advantage that the numerical computations could be performed much faster. In numerical examples, it was shown to have many of the desirable properties, in particular, both image intensity and the direction of edges are continued smoothly across the inpainting region. Their model was generalized for grayvalued images by Burger, He and Schönlieb [31].
Cahn-Hilliard inpainting for binary images can be thought of as a superposition of two gradient flows. The model reads: Let $f : \Omega \to \mathbb{R}$ be a given binary image, and suppose that $D \subset \Omega$ is the inpainting domain. We want to find a steady state solution $u(x)$ of

$$u_t = -\Delta\Big(\varepsilon\Delta u - \frac{1}{\varepsilon}W'(u)\Big) + \lambda(x)(f-u) \tag{2.41}$$

where W' is the gradient of a double well potential and

$$\lambda(x) \;=\begin{cases} 0 & \text{if } x \in D, \\ \lambda_0 & \text{if } x \in \Omega\backslash D \end{cases}$$

and $u(x,t)$ satisfies $\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0$ on $\partial\Omega$.

In practice, inpainting is done first with a larger ε, which blurres edges by diffusion and results in a topological reconnection of shapes in the damaged region. In a second step, one continues with the image obtained from step 1, but uses a much smaller ε now, in order to sharpen the edges after reconnection.

Equation (2.41) is identical with the standard Cahn-Hilliard equation except for the second term on the right-hand side. This *fidelity term* is there to keep the solution constructed close to the given image $f(x)$

in the complement of the inpainting domain, where there is image information available.
This means also that, if we set $\lambda(x) = 0$, which means that we consider only the inpainting domain, we recover the standard Cahn-Hilliard equation and can say that solutions to (2.41) should evolve along the steepest descent of the energy (2.11)

$$E_\varepsilon[u] = \int_D \varepsilon |\nabla u|^2 + \frac{1}{\varepsilon} W(u) dx.$$

The mass conservation constraint that $\int u(x,t)dx$ must be constant in t can therefore be interpreted as the principle that "the solution preserves total image intensity".
Consequently, one of the two gradient flows appearing in Cahn-Hilliard inpainting is the usual H^{-1} flow of (2.11) in the inpainting domain. The other gradient flow is the L^2-gradient flow of the pointwise energy

$$\int_D (u - f)^2 \, dx \tag{2.42}$$

outside of the inpainting domain.
Note that though Cahn-Hilliard inpainting can be thought of as a superposition of two gradient flows, it is neither a H^{-1}-gradient flow nor a L^2-gradient flow for the sum of the energies (2.11) and (2.42).
We can then recall the results from the usual Cahn-Hilliard equation case, namely that an arbitrary initial datum will form interfaces of thickness approximately ε at a fast time scale; these interfaces separate regions where the solution is approximately either 0 or 1 (given that in the choice of W the location of the wells were suitable adjusted to these values).
The fact that the energy (2.11) is decreased in the inpainting region suggests that the subsequent evolution involves some sort of coarsening of this configuration of regions.
Also, the sharp interface limit as $\varepsilon \to 0$ was studied in the context of inpainting: In [22], it was shown that at a slower time scale the interfaces approximate the solution of the Mullins-Sekerka (Hele-Shaw) problem. In [31], however, it was shown that solutions of an appropriate time-discrete Cahn-Hilliard inpainting approach Γ-converge, as $\varepsilon \to 0$, to solutions of an optimisation problem regularized with the TV-Norm.

Chapter 3

Global existence of solutions for special initial data

In this chapter, we analyze a semidiscrete scheme for the Cahn-Hilliard equation in one space dimension, when the interface length parameter is equal to zero. We prove convergence of the scheme for a suitable class of initial data, and we identify the limit equation. We also characterize the long-time behavior of the limit solutions.
This chapter contains the published work [60], written in collaboration with Matteo Novaga.

keywords: nonconvex functionals, forward-backward parabolic equations, finite element method.

3.1 Introduction

Motivated by several models in phase transitions and image processing, Cahn-Hilliard type equations have been extensively studied in recent years. In one space dimension, a typical example of such equation is

$$u_t = \frac{1}{2}\left(W'(u_x)\right)_x \qquad \text{in } [0,1]\times[0,T], \tag{3.1}$$

where u_x is the derivative of a Lipschitz continuous, one-periodic function $u : [0,1] \to \mathbb{R}$ and W is the nonconvex energy density $W(p) = \frac{1}{4}(p^2-1)^2$ (double well potential). Equation (3.1) is the formal L^2-gradient flow of the functional

$$E[u] := \frac{1}{2}\int_0^1 W(u_x)\,dx. \tag{3.2}$$

Notice that, by the change of variables $v = u_x$, equation (3.1) reduces to

$$v_t = \frac{1}{2}\left(W'(v)\right)_{xx} \quad \text{in} \quad [0,1]\times[0,T], \tag{3.3}$$

which corresponds to the H^{-1}-gradient flow of (3.2). We point out that, due to the nonconvexity of W, equations (3.1) and (3.3) are not well-posed.

In this paper, we deal with the semidiscrete problem

$$\begin{aligned} \frac{du^h}{dt} &= D^+W'(D^-u^h) && \text{in } [0,1]\times[0,T] \\ u^h(\cdot,0) &= \overline{u}^h && \text{on } [0,1]\times\{0\} \end{aligned} \tag{3.4}$$

where $h > 0$ is the grid size, D^+, D^- are the difference quotients defined in Definition 3.2.1, and $\overline{u}^h$ is the discretization of a piecewise-smooth function with nondifferentiable points $a_1, \ldots, a_m$. We consider (3.4) coupled with the periodic boundary conditions

$$\begin{aligned} u^h(0,t) &= u^h(1,t) && \text{on } \{0,1\}\times[0,T] \\ D^-u^h(0,t) &= D^-u^h(1,t) && \text{on } \{0,1\}\times[0,T] \end{aligned} \tag{3.5}$$

In Proposition 3.2.10 we show that, if the initial datum satisfies

$$\frac{1}{\sqrt{3}} \le \left|D_h^-\overline{u}^h\right| \le \alpha := \frac{1}{2}\left(\sqrt{3}+\frac{1}{\sqrt{3}}\right) \qquad \text{in } \bigcup_{j=1}^{m-1}\left[a_j^h, a_{j+1}^h\right], \tag{3.6}$$

then this property holds for all times $t \ge 0$. This assumption guarantees that the backward-parabolic zone, which is unstable for the evolution, consists of a finite number of points where the derivative u_x jumps from a region of (local) convexity of W to the another. We believe that weakening assumption (3.6) is an important but difficult task, and may lead to new interesting phenomena. We refer to the end of Section 3.3.1 for a discussion of such issue.

The main result of this paper, proved in Section 3.3, is the convergence of solutions to (3.4) and (3.5), as $h \to 0$, for initial data satisfying (3.6). We point out that, even under this simplifying assumption on initial data, we could find in the literature three different notion of solution to (3.1), which we briefly review for the reader's convenience.

An important consequence of this work is that the regularization of (3.1) by means of a spatial semidiscrete scheme produces in the limit a solution which coincides to the one proposed by Plotnikov in [104], and further analyzed by Evans and Portilheiro in [49].
There is no classical theory for solutions of forward-backward parabolic equations like (3.1) and (3.3), a part from some results on special solutions (see for instance [62] and references therein). However, several notions of weak solution have been proposed:

(i) In [44] the author defines an implicit variational scheme for the functional (3.2) which gives in the limit a solution to

$$u_t = \frac{1}{2}\left(W^{**\prime}(u_x)\right)_x \qquad \text{in } [0,1]\times[0,T],$$

where W^{**} is the convexified potential

$$W^{**} = \max\{f \leq W : \; f \text{ is convex}\}.$$

(ii) In [41] the following fourth-order regularization of (3.1) is considered:

$$u_t = -\varepsilon u_{xxxx} + W'(u_x)_x \tag{3.7}$$

and the author conjectures the existence of a pointwise limit as $\varepsilon \to 0$. The dynamics of this regularization for small ε, which is quite involved and has at least three relevant scales, was studied in [109, 14], where the asymptotic behavior as $t \to \infty$ is also discussed.

(iii) In [104] the author considers the regularization

$$u_t = \varepsilon u_{txx} + W'(u_x)_x\,, \tag{3.8}$$

proving the convergence, as $\varepsilon \to 0$, to a measure valued solution to (3.1). In [49] further properties of such limit solutions are discussed, with particular emphasis on a hysteresis phenomenon which also appears in our scheme.

Our approach is different from the ones mentioned above: instead of studying continuous regularizations, we perform a spatial semidiscretisation using the standard finite element method. In Section 3.2, we discuss the properties of the Cauchy problem for the semidiscrete scheme (see (3.13)) and provide suitable assumptions on initial data under which solutions are stable. One expects convergence of the scheme to classical solutions of (3.1) at least when the gradient of the initial datum takes values in the forward parabolic region, and we confirm such expectation with the only restriction that the gradient is not too big (see (3.6)). This is an advantage with respect to variational methods like the implicit scheme discussed in [44], which selects a local minimum of (3.2) and automatically forces all 1-Lipschitz functions not to move. Convergence of the scheme as the grid size h goes to zero is proved in Section 3.3, where we also identify the limit problem. We point out that our limit problem coincides with the limit of the continuous regularization (3.8), but not with the regularization (3.7). Finally, in Section 3.3.2 we prove the existence of a unique asymptotic state of the solution u, as $t \to +\infty$, whose derivative assumes precisely two values.

In order to keep the focus on the analytical aspects of the problem, we will not discuss the optimal convergence rate of the scheme, nor provide numerical simulations. We address the interested reader to [14, 46] for numerical simulations in the one-dimensional case, or to [52] for higher dimensions. A finite element discretisation of a simplified granular material model related to (3.1) was performed in [118] (see also [47]), where the authors study the limit profiles as $t \to +\infty$ of the discrete solutions.

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3.2 Spatial semidiscretisation

Let $\mathrm{I} := [0, 1]$ and let $\{h, \ldots, N\,h\}$ be a uniform grid on I with grid size $h = 1/N$, where $N \in \mathbb{N}$. Since we will work with 1-periodic functions, we identify the node 0 with the node N, hence $N + i$ with i. We denote by $PL(\mathrm{I})$ the N-dimensional vector subspace of $W^{1,\infty}(\mathrm{I})$, consisting of all continuous functions $u : \mathrm{I} \to \mathbb{R}$, with $u(0) = u(1)$, which are linear on the intervals $((i-1)h, ih)$, for all $i \in \{1, \ldots, N\}$. We also let $PC(\mathrm{I})$ be the N-dimensional vector subspace of $L^2(\mathrm{I})$ of all right-continuous piecewise-constant functions on the grid. Letting $u_i := u(ih)$, we can identify $u \in PL(\mathrm{I})$ (resp. $u \in PC(\mathrm{I})$) with the vector $u^h := (u_1, u_2, \ldots, u_N)$. Both $PL(\mathrm{I})$ and $PC(\mathrm{I})$ are endowed with the norms

$$\|u^h\|_{L^\infty(\mathrm{I})} := \max\{|u_i| : i = 1, \ldots, N\} \qquad \|u^h\|^2_{L^2_h(\mathrm{I})} := h \sum_{i=1}^{N} u_i^2 .$$

Notice that $\|u^h\|_{L^2_h(\mathrm{I})} = \|u^h\|_{L^2(\mathrm{I})}$ for all $u \in PC(\mathrm{I})$, and

$$\|u^h\|_{L^2(\mathrm{I})} \leq \|u^h\|_{L^2_h(\mathrm{I})} \leq \sqrt{3}\, \|u^h\|_{L^2(\mathrm{I})} \qquad \forall u \in PL(\mathrm{I}) . \tag{3.9}$$

Definition 3.2.1. We define the map $D^- : PL(\mathrm{I}) \to PC(\mathrm{I})$ and its adjoint $D^+ : PC(\mathrm{I}) \to PL(\mathrm{I})$ as

$$(D^- u^h)_i = \frac{u_i - u_{i-1}}{h} \qquad (D^+ w)_i = \frac{w_{i+1} - w_1}{h} \qquad i \in 1, \ldots, N.$$

With this notation, the space discretisation of (3.1) can be expressed by the following system of ODEs on $PL(\mathrm{I})$:

$$\begin{aligned} \frac{du_i}{dt} &= -\frac{1}{h}\frac{\partial \mathcal{F}}{\partial u_i} = \frac{1}{h}\left(W'\left(\frac{u_{i+1}-u_i}{h}\right) - W'\left(\frac{u_i - u_{i-1}}{h}\right)\right) \\ &= (D^+ W'(D^- u))_i , \end{aligned} \tag{3.10}$$

for all $i \in \{1, \ldots, N\}$, with periodic boundary conditions.
We now introduce the class of initial data for (3.1) which we will consider in this paper.

Assumption 3.2.1. *Let* $\{a_j\}_{j=1}^m \in (0,1)$, *with* $a_1 < a_2 < \ldots < a_m$. *We shall consider initial* $\overline{u} \in W^{1,\infty}(\mathrm{I}) \cap C^1(\mathrm{I} \setminus \{a_1, \ldots, a_m\})$ *such that* $\overline{u}(0) = \overline{u}(1)$ *and* $\overline{u}_x(0) = \overline{u}_x(1)$.

Remark 3.2.2. Notice that, if u solves

$$\begin{aligned} u_t &= W'(u_x)_x && \text{in } \mathrm{I} \times [0, +\infty) \\ u(0,t) &= u(1,t) && \text{on } \partial\mathrm{I} \times [0, +\infty) \\ u_x(0,t) &= u_x(1,t) && \text{on } \partial\mathrm{I} \times [0, +\infty), \end{aligned} \tag{3.11}$$

then $v = u_x$ solves

$$\begin{aligned} v_t &= W'(v)_{xx} && \text{in } \mathrm{I} \times [0, +\infty) \\ v(0) &= v(1) && \text{on } \partial\mathrm{I} \times [0, +\infty) \\ v_x(0,t) &= v_x(1,t) && \text{on } \partial\mathrm{I} \times [0, +\infty). \end{aligned} \tag{3.12}$$

Conversely, if $v = u_x$ solves (3.12) and $\int_\mathrm{I} v\, dx = 0$, then u solves (3.11). To get the full equivalence, i.e. for $\int_\mathrm{I} v\, dx = c$, it is enough to substitute the second line in (3.11) with $u(0,t) = u(1,t) + c$. For simplicity of the presentation, we restrict to the case $c = 0$.

Assumption 3.2.2. *Let $\overline{u}$ be as in Assumption 3.2.1. We denote by $a_1^h, \ldots, a_m^h$ be the grid points corresponding to the nondifferentiable points of $\overline{u}$, that is, $a_i \in [a_i^h, a_i^h + h)$ for all $i \in \{1, \ldots, N\}$. For the discrete initial data $\overline{u}^h \in PL(\mathrm{I})$ we require*

$$\|\overline{u}^h - \overline{u}\|_{L^\infty(\mathrm{I})} \underset{h\to 0}{\longrightarrow} 0 \; ; \quad \|D^-\overline{u}^h - \overline{u}_x\|_{L^1(\mathrm{I})} \underset{h\to 0}{\longrightarrow} 0 \; ; \quad \|D^-\overline{u}^h\|_{L^\infty(\mathrm{I})} \le C,$$

for some $C > 0$ independent of h.

The Cauchy problem corresponding to (3.10) is

$$\begin{aligned} \frac{du^h}{dt} &= D^+W'(D^-u^h) && \text{in } \mathrm{I} \times [0,T] \\ u^h(0,t) &= u^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T] \\ D^-u^h(0,t) &= D^-u^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T] \\ u^h(\cdot,0) &= \overline{u}^h && \text{on } \mathrm{I} \times \{0\}. \end{aligned} \tag{3.13}$$

where $\overline{u}^h \in PL(\mathrm{I})$ denotes the discrete initial datum with the properties listed in Assumption 3.2.2. Note that, due to the smoothness of W, the scheme (3.13) admits a unique solution $u^h \in C^\infty([0,t_0], PL(\mathrm{I}))$ for a suitable $t_0 > 0$. Moreover, by direct integration we get

$$\int_{\mathrm{I}} u^h(x,t)\,dx = \int_{\mathrm{I}} \overline{u}^h(x)\,dx. \tag{3.14}$$

In many cases, it will be useful to work with the system governing the evolution of the spatial derivative of $u^h(x,t)$.

Proposition 3.2.3. *Let $\overline{u}^h \in PL(\mathrm{I})$ be a discrete initial datum for (3.13) satisfying Assumption 3.2.2. If $u^h(x,t)$ is a solution to the Cauchy problem* (3.13)*, then $v^h := D^-u^h$ is a solution to the following system of ODEs:*

$$\begin{aligned} \frac{dv^h}{dt} &= D^-D^+W'(v^h)\,, && \text{in } \mathrm{I} \times [0,T] \\ v^h(0,t) &= v^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T] \\ D^-v^h(0,t) &= D^-v^h(1,t) && \text{on } \partial\mathrm{I} \times [0,T] \\ v^h(\cdot,0) &= D^-\overline{u}^h && \text{on } \mathrm{I} \times \{0\}. \end{aligned} \tag{3.15}$$

3.2.1 A priori estimates

We denote by $\alpha > 1$ the real number such that

$$W'(\alpha) = \alpha^3 - \alpha = W'\left(-\frac{1}{\sqrt{3}}\right),$$

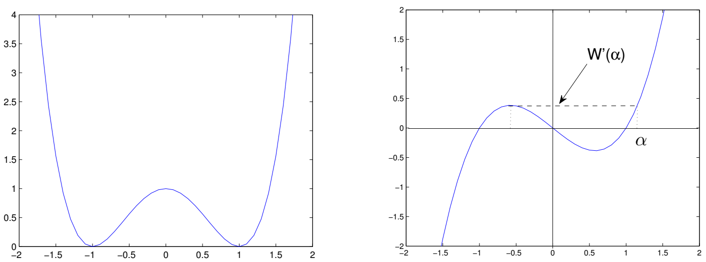

Figure 3.1: *At the left, the graph of the potential W; at the right, the graph of its derivative.*

see Figure 1. Let us denote by $M(t) := \max_{i=1,\dots,N} v_i(t)$ and $m(t) := \min_{i=1,\dots,N} v_i(t)$ the maximum and minimum of the nodal values of v, respectively.
The following result will be needed in Proposition 3.2.5; the proof can be found in [16, Lemma 5.1 and 5.2].

Lemma 3.2.4. *Let $v_1, \dots, v_N$ be real differentiable functions in an interval $[0,T)$. Define $M(t) := \max_{i=1,\dots,N} v_i(t)$. Then $M(t)$ is continuous, right-differentiable in $[0,T)$ and*

$$\frac{d}{dt^+}M(t) = \max_{i=1,\dots,N}\left\{\frac{d}{dt^+}v_i(t) \ : \ v_i(t) = M(t)\right\} \qquad \forall t \in [0,T).$$

Proposition 3.2.5 (L^∞ estimate)**.** *Let $u^h(t)$ be solutions of the discrete Cauchy problem (3.13) with initial data $\overline{u}^h$ satisfying Assumption 3.2.2. Then*

$$\|v^h(t)\|_{L^\infty(\mathrm{I})} \leq c \quad \forall t \in [0,\infty) \tag{3.16}$$

$$\|u^h(t)\|_{L^\infty(\mathrm{I})} \leq c \quad \forall t \in [0,\infty) \tag{3.17}$$

where the constant $c > 0$ is independent of h.

Proof. At time $t = 0$, the statement follows directly from the assumptions on the initial data.
Step 1. Let us first prove (3.16). We will show that $\max_i |v_i(t)|$ is nonincreasing whenever it is greater than α. We distinguish two cases.
Case 1: $\max_i |v_i(t)| = M(t) \geq \alpha$. As $M(t)$ is a solution to (3.15), by

Lemma 3.2.4 we have

$$\begin{aligned}\frac{d}{dt^+}M(t) &= \max_{i=1,\dots,N}\left\{\frac{d}{dt^+}v_i(t)\ :\ v_i(t)=M(t)\right\}\\ &= \max_{i:v_i(t)=M(t)} D^-D^+W'(v_i)\\ &= \frac{1}{h^2}\max_{i:v_i(t)=M(t)}\left(W'(v_{i+1})-2W'(M)+W'(v_{i-1})\right).\end{aligned}$$

From $M \geq \alpha$ and $M \geq v_{i\pm 1}$ we then get

$$W'(M(t)) \geq \max\{W'(v_{i-1}), W'(v_{i+1})\}.$$

Hence

$$\max_{i:v_i(t)=M(t)} \frac{dv_i}{dt^+} \leq 0, \tag{3.18}$$

which gives the upper bound

$$\max_{i=1,\dots,N} v_i(t) \leq \max_{i=1,\dots,N}\{\alpha, \max_i v_i(0)\}. \tag{3.19}$$

Case 2: $\max_i |v_i(t)| = -m(t) \geq \alpha$. Reasoning as above we obtain

$$\min_{i=1,\dots,N} v_i(t) \geq \min_{i=1,\dots,N}\{-\alpha, \min_i v_i(0)\}. \tag{3.20}$$

Putting together (3.19) and (3.20), we finally get

$$\|v^h(t)\|_{L^\infty(\mathrm{I})} \leq \max\{\alpha, \|D^-\overline{u}^h\|_{L^\infty(\mathrm{I})}\} \qquad \forall t \in [0,\infty)$$

which is (3.16).

Step 2. Estimate (3.17) now follows directly from (3.14) and (3.16).

□

Theorem 3.2.6 (Global existence of discrete solutions)**.** *Assume that the initial datum* $\overline{u}^h \in PL(\mathrm{I})$ *in (3.13) satisfies the periodic boundary conditions* $\overline{u}^h(0) = \overline{u}^h(1)$*. Then the Cauchy problem (3.13) admits a unique global solution* $u^h \in C^\infty([0,+\infty), PL(\mathrm{I}))$*.*

Proof. As we noted before, there exists a solution $u^h \in C^\infty([0,t_0], PL(\mathrm{I}))$, for some $t_0 > 0$. Proposition 3.2.5 guarantees L^∞-bounds on both $u^h(t)$ and its discrete derivative $v^h(t)$, which are uniform in time. As a consequence, the solution to the Cauchy problem can be extended for all times $t \in [0,+\infty)$. □

Proposition 3.2.7 (Energy decreasing property). *Let $u^h(x,t)$ be the solution of* (3.13) *with an initial datum $\overline{u}^h$ satisfying Assumption 3.2.2. Define the discrete energy*

$$E^h(t) := E[u^h(\cdot,t)] := h\sum_{i=1}^{N} W(D^- u_i(t)). \tag{3.21}$$

Then the following relation holds:

$$\frac{d}{dt}E[u^h(\cdot,t)] = -\|u_t^h\|^2_{L^2_h(\mathrm{I})} \le 0. \tag{3.22}$$

Proof. Keeping in mind the periodic boundary conditions, we compute

$$\begin{aligned}\frac{d}{dt}E[u^h(\cdot,t)] &= h\sum_{i=1}^{N} W'(D^- u_i)\partial_t(D^- u_i)\\ &+W'(D^- u_N)\partial_t u_N - W'(D^- u_0)\partial_t u_0\\ &= -\,h\sum_{i=1}^{N} D^+ W'(D^- u^h)_i\,\partial_t u_i\\ &= -\,\|u_t^h(\cdot,t)\|^2_{L^2_h(\mathrm{I})} \le 0.\end{aligned}$$

□

As a consequence, we have

$$E^h(t) \le E^h(0) \le C \qquad \forall t \ge 0, \tag{3.23}$$

as the discrete initial datum is bounded in $W^{1,\infty}(\mathrm{I})$ uniformly in h, by Assumption 3.2.2.

Corollary 3.2.8 (Hölder continuity). *Let $\overline{u}^h$ be initial data satisfying Assumption 3.2.2. Then the solutions u^h of* (3.13) *are uniformly bounded in $C^{\frac{1}{2}}\left([0,T);L^2_h(\mathrm{I})\right)$.*

Proof. Let $0 \le t_1 < t_2 < +\infty$. We have to show $\|u^h(t_1)-u^h(t_2)\|_{L^2_h} \le c\,|t_2-t_1|^{\frac{1}{2}}$, for some constant $c>0$ independent of h. Using Hölder inequality and (3.23), we get

$$\begin{aligned}\|u^h(t_1)-u^h(t_2)\|_{L^2_h(\mathrm{I})} &\le |t_2-t_1|^{\frac{1}{2}}\left(E^h(t_1)-E^h(t_2)\right)^{\frac{1}{2}}\\ &\le \sqrt{E^h(0)}\,|t_2-t_1|^{\frac{1}{2}}.\end{aligned}$$

□

The following corollary will be an important ingredient in the convergence proof.

Corollary 3.2.9. *Let* $u^h(x,t)$ *be solutions to the Cauchy problem* (3.13) *with initial data* $\overline{u}^h$ *satisfying Assumption 3.2.2. Then* $\frac{d}{dt}u^h \in L^2\big([0,\infty), L^2_h(\mathrm{I})\big)$, *i.e.*

$$\int_0^\infty \left\| \frac{d}{dt} u^h(\cdot,t) \right\|^2_{L^2_h(\mathrm{I})} dt \le E^h(0) \le c, \tag{3.24}$$

where the constant c *is independent of* h.

Proof. Recalling (3.22) we have

$$E^h(0) - E^h(t) = \int_0^t \left\| u^h_t(\cdot,\tau) \right\|^2_{L^2_h(\mathrm{I})} d\tau \qquad \forall t \ge 0\,.$$

As $E^h(0) \le C$ by Assumption 3.2.2, the thesis follows letting $t \to +\infty$. □

3.2.2 The stability estimate

We shall make another assumption on initial data $\overline{u}^h$ which guarantees the stability of the solution to (3.13): we take initial data $\overline{u} \in W^{1,\infty}(\mathrm{I})$ as in Assumption 3.2.1 which further satisfy

$$\frac{1}{\sqrt{3}} \;\le\; (-1)^{j+1}\,\overline{u}_x(x) \;\le\; \alpha \qquad \forall\, x \in (a_{j-1}, a_j)\;, \quad j \in \{1,\dots,m\}\,. \tag{3.25}$$

Note that (3.25) implies in particular that m is even and $\overline{u}_x$ takes values only in the regions where the potential W is convex.
We point out that a similar assumption was made in [17] for the Perona-Malik equation.
We now formulate the discrete analog of (3.25).

Assumption 3.2.3. *Let* α *as above and let* $\overline{u}^h \in PL(\mathrm{I})$ *be discrete initial data satisfying Assumption 3.2.2, with* $a^h_1, \dots, a^h_m$ *the grid points corresponding to the nondifferentiability points of* $\overline{u}$. *We require that* $\overline{u}^h$ *satisfies*

$$\frac{1}{\sqrt{3}} \;\le\; (-1)^{j+1}\, D^-\overline{u}_i \;\le\; \alpha \quad \forall\, ih \in \big(a^h_{j-1}, a^h_j\big]\;,\; j \in \{1,\dots,m\} \tag{3.26}$$

Proposition 3.2.10 (Stability estimate). *Let u^h be solutions to (3.13) with initial data $\overline{u}^h$ satisfying Assumptions 3.2.2 and 3.2.3. Then u^h satisfies*

$$\frac{1}{\sqrt{3}} \;\leq\; (-1)^{j+1}\, D^- u_i(t) \;\leq\; \alpha \quad \forall\, ih \in \left(a^h_{j-1}, a^h_j\right] \;,\; j \in \{1, \ldots, m\} \\ ,\, t \geq 0\,. \tag{3.27}$$

Proof. Fix $j \in \{1, \ldots, m\}$. Without loss of generality we can assume that $\overline{u}^h$ is monotone increasing on $[a^h_j h, a^h_{j+1} h]$, that is $v_i(0) = D^- \overline{u}_i \in [1/\sqrt{3}, \alpha]$ in $[a^h_j h, a^h_{j+1} h]$. We let

$$m(t) := \min_{i=1,\ldots,N} v_i(t) \qquad M(t) := \max_{i=1,\ldots,N} v_i(t)\,,$$

and distinguish two cases:
Case 1: $M(t) = \alpha$ for some $t \geq 0$. By Lemma 3.2.4 and (3.15) we have

$$\begin{aligned} \frac{d}{dt^+} M(t) &= \max_{i:\, v_i(t) = M(t)} \frac{d}{dt^+} v_i(t) \;=\; \max_{i:\, v_i(t) = M(t)} D^- D^+ W'(v_i) \\ &= \frac{1}{h^2} \max_{i:\, v_i(t) = M(t)} \left(W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})\right) \leq 0, \end{aligned} \tag{3.28}$$

where we used the fact that $W'(\alpha) \geq W'(x)$ for all $x \leq \alpha$.
Case 2: $m(t) = 1/\sqrt{3}$ for some $t \geq 0$. As above, we have

$$\begin{aligned} \frac{d}{dt^+} m(t) &= \min_{i:\, v_i(t) = m(t)} \frac{d}{dt^+} v_i(t) \;=\; \min_{i:\, v_i(t) = m(t)} D^- D^+ W'(v_i) \\ &= \frac{1}{h^2} \min_{i:\, v_i(t) = m(t)} \left(W'(v_{i+1}) - 2W'(M) + W'(v_{i-1})\right) \geq 0, \end{aligned} \tag{3.29}$$

where we used the fact that $W'(1/\sqrt{3}) \leq W'(x)$ for all $x \geq -\alpha$.
The thesis follows from (3.28) and (3.29). □

3.3 Convergence of the scheme

Proposition 3.3.1. *Let the initial data $\overline{u}^h$ satisfy Assumption 3.2.2. Then the solutions u^h converge, up to a subsequence as $h \to 0$, to a limit function $u \in C\,(\mathrm{I} \times [0, +\infty))$, uniformly on compact subset of $\mathrm{I} \times [0, +\infty)$.*

Proof. By (3.9), Proposition 3.2.5 and Corollary 3.2.9 we know that the solutions u^h are uniformly bounded in $X_T := H^1\left([0,T], L^2(\mathrm{I})\right) \cap L^\infty\left([0,T], W^{1,\infty}(\mathrm{I})\right)$, for all $T > 0$. The thesis follows form the compact embedding of X_T into $C\left(\mathrm{I} \times [0,T]\right)$ [17]. $\square$

Recalling Proposition 3.2.10 and reasoning exactly as in [61, Proposition 3.3], we obtain the following estimate.

Lemma 3.3.2. *Let $u^h(t)$ be a solution to the Cauchy problem* (3.13) *with initial data $\overline{u}^h$ satisfying Assumptions 3.2.2 and 3.2.3. Then, for every open set $\mathrm{I}_1 \subset\subset \mathrm{I} \setminus \{a_1, \dots a_m\}$, there exists a constant $c = c(\mathrm{I}_1)$ such that for h small enough there holds*

$$\left\| \frac{d}{dt} u^h(t) \right\|^2_{L^2_h(\mathrm{I}_1)} \leq\ E^h(0)\ \left(\frac{1}{t} + c \right) \qquad \forall t > 0. \tag{3.30}$$

Proposition 3.3.3. *Let $\overline{u}^h$ be initial data satisfying Assumptions 3.2.2 and 3.2.3, and let u^h be the corresponding solutions to the Cauchy problem (3.13). Then, for any compact subset K of $\mathrm{I} \setminus \{a_1, \dots, a_m\}$ and for every $t > 0$, there exists a function $\psi \in H^1(K)$ such that*

$$W'\left(v^h\right) \longrightarrow \psi \quad \text{uniformly on } K \text{ (up to a subsequence).}$$

Proof. As $W'(v^h)$ is uniformly bounded in $L^\infty(\mathrm{I})$ by Assumption 3.2.3, up to a suitable subsequence we have

$$W'\left(D^- u^h\right) \longrightarrow \psi \quad \text{weakly}^* \text{ in } L^\infty(K).$$

Moreover, by (3.9) and Lemma 3.3.2 $\frac{d}{dt} u^h = D^+ W'\left(D^- u^h\right)$ is uniformly bounded in $L^2(\mathrm{I})$. The thesis then follows from the Arzelà-Ascoli Theorem. $\square$

Proposition 3.3.3 allows us to obtain the strong convergence of $D^- u^h$, which is needed to pass to the limit in the nonlinear problem (3.13).

Proposition 3.3.4. *Let $u^h(t)$ be solutions to the Cauchy problem* (3.13) *with initial data $\overline{u}^h$ satisfying Assumptions 3.2.2 and 3.2.3. Then, up to a subsequence as $h \to 0$,*

$$D^- u^h \longrightarrow u_x \qquad \text{a.e. on } \mathrm{I} \times [0, +\infty) \tag{3.31}$$

and

$$W'\left(D^- u^h\right) \longrightarrow W'\left(u_x\right) \text{ in } L^2_{\mathrm{loc}}(\mathrm{I} \times [0, +\infty)). \tag{3.32}$$

Proof. By Propositions 3.2.5 and 3.3.1 we have

$$D^- u^h \longrightarrow u_x \quad \text{weakly}^* \text{ in } L^\infty(\mathrm{I}) \text{ for every } \; t \geq 0. \tag{3.33}$$

Let K be as in Proposition 3.3.3. As W' is invertible on $[-\alpha, -1/\sqrt{3}]$ and $[1/\sqrt{3}, \alpha]$, Proposition 3.3.3 implies

$$D^- u^h(t) = (W')^{-1}\left(W'(D^- u^h(t))\right) \longrightarrow u_x(t) \quad \text{uniformly on } K$$

for all $t > 0$, which gives (3.31). Claim (3.32) then follows from (3.31) and Lebesgue's Theorem. $\square$

3.3.1 The limit problem

Theorem 3.3.5. *Let $\overline{u} \in W^{1,\infty}(\mathrm{I})$ be an initial datum satisfying Assumptions 3.2.1 and (3.25). Let $\overline{u}^h$ be finite element discretizations of $\overline{u}$ satisfying Assumptions 3.2.2 and 3.2.3, let u^h be the corresponding solutions to (3.13), and let $u \in C(\mathrm{I} \times [0, +\infty))$ be the limit of u^h, as $h \to 0$, given by Proposition 3.3.1. Then u is the unique solution to the following PDE:*

$$\begin{array}{lll} (i) & u_t = W'(u_x)_x & \text{in } (\mathrm{I} \setminus \{a_1, \ldots, a_m\}) \times [0, +\infty) \\ (ii) & W'(u_x^-) = W'(u_x^+) & \text{on } \{a_1, \ldots, a_m\} \times [0, +\infty) \\ (iii) & u^- = u^+ & \text{on } \{a_1, \ldots, a_m\} \times [0, +\infty) \\ (iv) & u(0) = \overline{u} & \text{at } \mathrm{I} \times \{0\}, \end{array} \tag{3.34}$$

where we set

$$u^\pm := \lim_{x \to a_j^\pm} u(x) \qquad u_x^\pm := \lim_{x \to a_j^\pm} u_x(x).$$

In particular $W'(u_x) \in C(\mathrm{I} \times [0, +\infty))$ and
$u \in C^\infty\left((\mathrm{I} \setminus \{a_1, \ldots, a_m\}) \times (0, +\infty)\right)$.

Proof. Multiplying by $\varphi \in C_0^1(\mathrm{I} \times [0, +\infty))$ the first equation in (3.13), after an integration by parts we get

$$\int_0^\infty \int_{\mathrm{I}} u^h \, \varphi_t \, dx \, dt = \int_0^\infty \int_{\mathrm{I}} W'(D^- u^h) \, D^- \varphi \, dx \, dt. \tag{3.35}$$

As $u^h \to u$ locally uniformly on $\mathrm{I} \times [0, +\infty)$ by Proposition 3.3.1, and by Proposition 3.3.4 also $W'(D^- u^h) D^- \varphi \to W'(u_x)\varphi_x$ in $L^2(\mathrm{I} \times [0, +\infty))$, we can pass to the limit in (3.35):

$$\int_0^\infty \int_{\mathrm{I}} u \, \varphi_t \, dx \, dt = \int_0^\infty \int_{\mathrm{I}} W'(u_x) \, \varphi_x \, dx \, dt. \tag{3.36}$$

Since $u_t \in L^2(\mathrm{I} \times [0,+\infty))$, (3.36) imples $W'(u_x) \in L^2\left([0,+\infty), H^1(\mathrm{I})\right)$, so that we can integrate by parts and obtain

$$\int_0^\infty \int_{\mathrm{I}} u_t \varphi \, dxdt = \int_0^\infty \int_{\mathrm{I}} W'(u_x)_x \, \varphi \, dxdt, \tag{3.37}$$

which proves statement (i).
Equalities (ii) and (iii) follow from the continuity of $W'(u_x)$ and u, respectively. □

Remark 3.3.6. Problem (3.38) is equivalent to the limit problem derived in [104, 49] for the regularization (3.8). On the other hand, due to the numerical simulations performed in [14], it is expected to be different form the limit problem corresponding to the Cahn-Hilliard regularization (3.7) discussed in [41, 109].

Corollary 3.3.7. *If u satisfies (3.34), then $v = u_x = \lim_{h\to 0} v^h$ is the unique solution to the following PDE:*

$$\begin{aligned} v_t &= W'(v)_{xx} && \text{in } (\mathrm{I} \setminus \{a_1,\dots,a_m\}) \times [0,+\infty) \\ W'(v^-) &= W'(v^+) && \text{on } \{a_1,\dots,a_m\} \times [0,+\infty) \\ W'(v)^-_x &= W'(v)^+_x && \text{on } \{a_1,\dots,a_m\} \times [0,+\infty) \\ v(0) &= \overline{u}_x && \text{on } (\mathrm{I} \setminus \{a_1,\dots,a_m\}) \times \{0\}. \end{aligned} \tag{3.38}$$

Passing to the limit in (3.24) as $h \to 0$, we obtain an integral estimate on the time derivative of u.

Proposition 3.3.8. *Let u be as in Theorem 3.3.5. We have $u_t \in L^2\big(\mathrm{I} \times (0,\infty)\big)$ and*

$$\int_{\mathrm{I}\times(0,\infty)} \left(\frac{du}{dt}(x,t)\right)^2 dxdt = E[\overline{u}].$$

Let us briefly discuss the new phenomena which may occur if one tries to weaken Assumption (3.25). We first observe that the energy balance condition (3.34) (ii) must be satisfied by any limit of the semidiscrete scheme.

(i) If the lower bound $|\bar{u}_x| \geq 1/\sqrt{3}$ is violated, there are intervals of the domain I where the derivative of the solution belongs to the nonconvex region of the potential W. This leads to instability and one expects the onset of a microsctructure, due to rapid oscillations of the derivative. However, differently from other regularizations, such oscillations do not seem able to stop the evolution [16].

(ii) If the upper bound $|\bar{u}_x| \leq \alpha$ is violated, then also the lower bound cannot hold for positive times, due to the energy balance condition (3.34) *(ii)*. In this case one expects that the jump points of the derivative a_j move in time and a hysteresis phenomenon occurs, as discussed in [49].

3.3.2 Long-time behaviour

Theorem 3.3.9. *Let u be a solution of* (3.34)*. Then there exists a unique limit*

$$u_\infty(x) := \lim_{t\to+\infty} u(t,x) \qquad x \in \mathrm{I},$$

which is given by the piecewise-linear solution to

$$\begin{array}{lll} (i) & W'\left((u_\infty)_x\right)_x = 0 & \text{in } \mathrm{I} \setminus \{a_1, \dots, a_m\} \\ (ii) & W'\left((u_\infty)_x^-\right) = W'\left((u_\infty)_x^+\right) & \text{on } \{a_1, \dots, a_m\} \\ (iii) & u_\infty^- = u_\infty^+ & \text{on } \{a_1, \dots, a_m\}. \end{array} \tag{3.39}$$

Proof. We divide the proof into three steps.

Step 1 (Existence of u_∞). By Proposition 3.3.8, there exists a sequence of times $t_n \to +\infty$ such that

$$\int_{t_n}^{t_n+1} \|u_t\|^2_{L^2(\mathrm{I})}\, dt \;\longrightarrow\; 0. \tag{3.40}$$

We now define a sequence w^n of solutions to (3.34) in the following way:

$$w^n(x,t) \;:=\; u(x, t_n + t) \quad t \in [0,1].$$

From (3.40) we have

$$\int_0^1 \|w_t^n\|^2_{L^2(\mathrm{I})} dt \underset{n\to\infty}{\longrightarrow} 0, \tag{3.41}$$

whence $w^n \to w \in H^1\left([0,1], L^2(\mathrm{I})\right) \cap L^\infty([0,1], W^{1,\infty}(\mathrm{I}))$, with $w_t \equiv 0$, that is the limit function $w = u_\infty$ does not depend on t.

Step 2 (Limit equation). As every w^n solves (3.34), from (3.36) we get

$$\int_0^1 \int_{\mathrm{I}} W'(w_x^n) \varphi_x \, dx dt \;=\; 0,$$

for all test functions $\varphi \in C^1(\mathrm{I})$ independent of t. Passing to the limit as $n \to \infty$ and recalling (3.41), we get *(i)* and *(ii)*, while *(iii)* follows from the Lipschitz continuity of w.

We now show that w_x is a piecewise-constant function which assumes exactly two values, p^- and p^+. Indeed, (3.39) (i) implies that, for all $j \in \{1, \dots, m\}$, there exists $p_j \in [-\alpha, -1/\sqrt{3}] \cup [1/\sqrt{3}, \alpha]$ such that $W'(w_x) \equiv p_j$. Moreover, from condition (ii) we have that

$$W'(p_i) = W'(p_j) \qquad \forall\, i,\, j \in \{1, \dots, m\}\,. \tag{3.42}$$

Since W' is monotone in the intervals $[-\alpha, -1/\sqrt{3}]$ and $[1/\sqrt{3}, \alpha]$, we get that for all $p \in [1/\sqrt{3}, \alpha]$ there exists only one value $\tilde{p} \in [-\alpha, -1/\sqrt{3}]$ such that

$$W'(p) \;=\; W'(\tilde{p}). \tag{3.43}$$

The claim then follows from (3.42) and (3.43).

Step 3 (Uniqueness). Once we know that w_x assumes precisely two values $p^- < p^+$, with $p^- \in [-\alpha, -1/\sqrt{3}]$ and $p^+ \in [1/\sqrt{3}, \alpha]$, the uniqueness of such values follows by direct integration. More precisely, assuming without loss of generality $w_x = p^+ > 0$ on $[0, a_1]$ and recalling (3.39) (iii), we have

$$0 = w(1) - w(0) = \sigma(p^+), \tag{3.44}$$

where

$$\sigma(p) \;:=\; p \sum_{\ell=0}^{\frac{m}{2}-1} (a_{2\ell+1} - a_{2\ell}) + \tilde{p} \sum_{k=1}^{\frac{m}{2}} (a_{2k} - a_{2k-1}) \qquad p \in \left[\frac{1}{\sqrt{3}}, \alpha\right].$$

Since σ is strictly increasing on $[1/\sqrt{3}, \alpha]$, equation (3.44) uniquely determines the value of p^+, and consequently of p^-. $\square$

Chapter 4

Existence of solutions for a more general class of initial data

In this chapter, we study the convergence of a semidiscrete scheme for the forward-backward parabolic equation $u_t = (W'(u_x))_x$ with periodic boundary conditions in one space dimension, where W is a standard double-well potential. We characterize the equation satisfied by the limit of the discretized solutions as the grid size goes to zero. Using an approximation argument, we show that it is possible to flow initial data $\overline{u}$ having regions where $\overline{u}_x$ falls within the concave region $\{W'' < 0\}$ of W, where the backward character of the equation manifests. It turns out that the limit equation depends on the way we approximate $\overline{u}$ in its unstable region.

This chapter contains the published work [15], written in collaboration with Giovanni Bellettini and Matteo Novaga.

4.1 Introduction

In this paper we are interested in the existence of solutions to the gradient flow of the nonconvex and nonconcave functional

$$F(u) := \int_{\mathbb{T}} W(u_x)\, dx, \qquad W(p) := \frac{1}{4}(1-p^2)^2, \tag{4.1}$$

where $\mathbb{T}$ is the one-dimensional torus. The formal L^2-gradient flow of (4.1) leads to the forward-backward parabolic equation

$$u_t = (W'(u_x))_x \qquad \text{in } \mathbb{T} \times [0, +\infty), \tag{4.2}$$

that we couple with the initial condition

$$u(0) = \overline{u}.$$

As (4.2) is not well-posed due to the nonconvexity of W, it may fail to admit local in time classical solutions, at least for a large class of initial data $\overline{u}$. A typical source of instability is, for example, the case when there are intervals $I \subset \mathbb{T}$ for which

$$\overline{u}_x(x) \in (p^-, p^+), \qquad x \in I, \tag{4.3}$$

where

$$(p^-, p^+) = \{W'' < 0\} \tag{4.4}$$

is the concave region of W (in our case $p^- = -1/\sqrt{3}$ and $p^+ = 1/\sqrt{3}$). Indeed, under these conditions the backward character of the equation manifests and instabilities, such as the quick formation of microstructures, are expected, making the subsequent evolution difficult to describe.

It is therefore natural to approximate (4.2) with a regularized equation, and indeed different regularizations have been proposed in the literature. We recall in particular [104], where the author considers the regularization

$$u_t = \varepsilon u_{txx} + (W'(u_x))_x \,,$$

and proves convergence as $\varepsilon \to 0^+$ to a measure-valued solution to (4.2). In [49] (see also [110, 111] and references therein) further properties of such limit solutions are discussed.

We also mention that in [41] a fourth order regularization for another forward-backward parabolic equation (the Perona-Malik equation) is suggested, which in our case reads as

$$u_t = -\varepsilon u_{xxxx} + (W'(u_x))_x \,. \tag{4.5}$$

The dynamics of this regularization as $\varepsilon \to 0^+$ is quite involved and was studied in [109, 14, 12] (notice that setting $u_x = v$, equation (4.5) becomes the Cahn-Hilliard equation by differentiation). In particular, in [12] the authors are able to pass to the limit in (4.5) as $\varepsilon \to 0^+$, under the assumption that the initial data (which now depend also on ϵ) converge to $\overline{u}$ in a suitable energetic sense.

Another possible way to regularize (4.2) is by approximation with a semidiscrete scheme (see [60]), and this is the approach that we shall adopt in this paper. More precisely, let us consider the semidiscrete scheme

$$\begin{cases} \dfrac{du^h}{dt} = D_h^+ \, W'(D_h^- u^h) & \text{in } \mathbb{T} \times [0, +\infty), \\ \\ u^h(0) \, = \, \overline{u}^h & \text{on } \mathbb{T}, \end{cases} \tag{4.6}$$

where $h > 0$ denotes the grid size on the torus $\mathbb{T}$, $D_h^\pm$ are the difference quotients defined in (4.12), and $\overline{u}^h$ are the piecewise linear discrete initial data, with $\overline{u}^h \to \overline{u}$ in $L^\infty(\mathbb{T})$ as $h \to 0^+$.

The spatial discretization acts as a regularization of the ill-posed problem (4.2), and regular solutions u^h to (4.6) indeed exist for all times. Moreover, it has been shown in [60] that one can pass to the (full) limit in such solutions, as $h \to 0^+$, when the gradients of $\overline{u}$ and $\overline{u}^h$ lie in the region $\{W'' \geq 0\}$ (namely, the complement of the interval $(-1/\sqrt{3}, 1/\sqrt{3})$ appearing in (4.4), compare Figure 4.1), with possible jumps from one connected component to another[1] (see also [17, 61]).

The aim of this paper is to show convergence of (4.6) for a larger class of initial data, namely those satisfying

$$|\overline{u}_x(x)| \leq M^+ := 2/\sqrt{3}, \qquad x \in \mathbb{T}, \tag{4.7}$$

(see (4.10) for the definition[2] of M^+. In particular, we allow for initial data $\overline{u}$ satisfying (4.3). We believe this to be a significant improvement with respect to the previous results, for the above mentioned reason that these data quickly lead to instabilities and formation of microstructures.

One of the main ideas of the present paper is to consider the initial datum $\overline{u}$ endowed with an auxiliary function $\overline{\varrho} \in L^\infty(\mathbb{T}; [0, 1])$ measuring the percentage of mesh intervals where $\overline{u}^h$ is decreasing in a neighborhood of the point $x \in \mathbb{T}$, see Definition 4.4.1. The functions $\overline{u}$ and $\overline{\varrho}$ are not independent, since they must satisfy the compatibility condition (4.27). The reason for introducing $\overline{\varrho}$ is that it allows to construct a sequence $(\overline{u}^h)$ (depending on $\overline{\varrho}$) and converging to $\overline{u}$ as

[1] The solution found in [60] is such that $W'(u_x(\cdot, t))$ is continuous at the points where there is a jump from one connected component to another, namely where $u_x(\cdot, t)$ has a discontinuity, and these points, differently to what is expected [14] from the limits of (4.5), do not move in the horizontal direction.

[2] As we shall see, assumption (4.7) is necessary in order to approximate $\overline{u}$ with discrete initial data $\overline{u}^h$ such that the solutions u^h have discrete gradients $D_h^- u^h$ lying everywhere in the region $\{W'' \geq 0\}$ for all times $t \geq 0$.

$h \to 0^+$, with the crucial property that the discrete gradients $D_h^- \overline{u}^h$ always belong to $\{W'' \geq 0\}$ (see Section 4.4.2). This requirement is important since, under assumption (4.7), such a property is preserved by the semidiscrete scheme (4.6), as stated in (4.20). On the other hand, it is clear that this amounts to a restriction on the sequence of approximating initial data $\overline{u}^h$ that we are able to consider. We observe however that the requirement $D_h^- \overline{u}^h(x) \in \{W'' \geq 0\}$ for any $x \in \mathbb{T}$, is rather natural: indeed, given any initial condition $\overline{u}$, numerical experiments [53, 14] give evidence that the slope of the discrete solution, due to the quick formation of microstructures, takes values in this region after a very short time.
Summarizing, we can say that our initial conditions can be represented by a pair $(\overline{u}, \overline{\varrho})$ verifying (4.27). Observe that, given a function $\overline{u}$ satisfying (4.7), there always exists a function $\overline{\varrho}$ satisfying the required assumptions. We also notice that the function $\overline{\varrho}$ is not uniquely determined in general.
Under these hypotheses, we show in Theorem 4.6.7 that the sequence (u^h) of solutions to the semidiscrete scheme has enough compactness properties to pass to the limit. This crucial compactness cannot, however, be obtained for the sequence $(D_h^- u^h)$ of discrete gradients. Indeed, due to the instability of (4.2), u^h have oscillations which are typically of order h when $D_h^- u^h$ belongs to the nonconvex region of W, so that there is no hope to have strong convergence of $D_h^- u^h$ as $h \to 0^+$. Rather, it is possible to obtain a compactness property for $W'(D_h^- u^h)$: indeed, in Proposition 4.5.3 and Corollary 4.5.4 we show a uniform Hölder estimate for the sequence $\left(W'\left(D_h^- u^h\right)\right)$. However, these estimates are not enough for passing to the limit in the nonlinear term of the equation; the crucial point is then to gain a compactness property for the *averaged discrete gradients*

$$D_{n_i h}^- u^h \tag{4.8}$$

on an intermediate grid of size $n_i h$, where n_i is a suitable positive integer related to the values of $\overline{\varrho}$: see Definition 4.4.6 for the details. Therefore, we can say that introducing the function $\overline{\varrho}$ allows to identify an intermediate (or mesoscopic) scale, which in turn permits to obtain a compactness for the averaged gradients (4.8). Our conclusion (Theorems 4.6.9 and 4.6.14) is that the function $u := \lim_h u^h$ solves distributionally the limit problem

$$u_t = \left(W'\left(q\left(u_x, \overline{\varrho}\right)\right)\right)_x \tag{4.9}$$

where the map q, which depends on $\overline{\varrho}$, is defined in Definition 4.3.6, and is related to the two stable branches of the local inverse of W'.

No extraction of a subsequence is necessary in Theorems 4.6.9 and 4.6.14 (see Remark 4.6.10). We observe also that this result can be considered as a characterization of a (nonunique) selection among the infinitely many Young measure solutions to (4.2), obtained through the semidiscrete scheme (4.6).
The case when $\overline{u}$ is such that $\overline{u}_x(x) \in \{W'' \geq 0\}$ for any $x \in \mathbb{T}$ can be covered by taking either $\overline{\varrho} \equiv 0$ or $\overline{\varrho} \equiv 1$ in $\mathbb{T}$, and equation (4.9) reduces to equation (4.2) (well posed, in this situation)[3]. The particular case considered in [60] corresponds to a choice of $\overline{\varrho} \in L^\infty(\mathbb{T}; \{0,1\})$, which is possible only if the initial datum $\overline{u}$ avoids the nonconvex region (p^-, p^+) of W.

One may expect that a generic sequence of initial data $\overline{u}^h$ defines, up to subsequences, a limit function $\overline{u}$ and a function $\overline{\varrho} \in L^\infty(\mathbb{T}; [0,1])$, such that the corresponding function $u := \lim_h u^h$ is a solution of (4.9) with initial datum $\overline{u}$. Such an extension would require significant modifications to our proofs, and goes beyond the scope of this paper. However, we believe that it is already interesting to consider a specific choice of $\overline{\varrho}$, since it gives rise to a well-defined solution to a limit evolution problem, for a large class of initial data.
We stress that our result shows that the evolution law obtained as a limit of the semidiscrete scheme is not unique in general, and depends on the sequence $(\overline{u}^h)$ of initial data chosen to approximate $\overline{u}$. We also notice that there does *not* seem to be a choice of $\overline{\varrho}$ which reproduces the solution numerically observed in [14], for the limit of equation (4.5) as $\epsilon \to 0$.
Up to minor modifications, the results of this paper still hold if we replace W in (4.1) with any smooth enough double-well potential with similar qualitative (in particular convexity/concavity) properties.

The plan of the paper is the following. In Section 4.2 we introduce the map T, leading to the definition of the function

$$q(p, \sigma)$$

used in (4.9), compare also (4.28). In Section 4.3 we introduce the semidiscrete scheme (4.6) for equation (4.2). In Section 4.4 we define the class of initial data for which we are able to show convergence of the semidiscrete scheme. In Section 4.5 we prove the apriori estimates which give compactness of the discrete solutions. Finally, in Section 4.6 we pass to the limit in the discretized equation (4.6) and characterize the limit evolution law. For simplicity of presentation, the proof

[3]Notice that also in this case, which is the simplest possible, since there are infinitely many choices of $\overline{\varrho}$ (different from $\overline{\varrho} \equiv 0$ or $\overline{\varrho} \equiv 1$), one gets infinitely many possibile different limit equations.

will be given first for a piecewise constant function $\overline{\varrho}$ taking rational values in $[0,1]$, and then for a general $\overline{\varrho}$.

4.2 Unstable slopes and the map T

In this section we introduce a map T which will be of crucial importance in our notion of solution to equation (4.2).
Recall our notation: we have $W'(p) = p^3 - p$, and the concave region of W is the interval

$$(p^-, p^+) := \{p \in \mathbb{R} : W''(p) < 0\},$$

which represents the set of "unstable" slopes. In our specific case we have

$$-p^- = p^+ = \frac{1}{\sqrt{3}}.$$

If $q \in (p^-, p^+)$ then $W'^{-1}(W'(q))$ consists of three distinct elements (one belonging to the unstable branch of the local inverse of W', and the other two belonging to the stable branches), and if $q \in \{p^-, p^+\}$ then $W'^{-1}(W'(q))$ reduces to two distinct elements, that we write as follows:

$$W'^{-1}\big(W'(p^-)\big) = \{p^-, M^+\}, \qquad W'^{-1}\big(W'(p^+)\big) = \{p^+, M^-\}, \tag{4.10}$$

where $p^- < M^+$ and $M^- < p^+$, see Figure 4.1. With our choice of W, we have $W'(p^-) = -W'(p^+) = \frac{2}{3\sqrt{3}}$,

$$-M^- = M^+ = \frac{2}{\sqrt{3}}.$$

We now introduce the map T on the disconnected set $[M^-, M^+] \setminus (p^-, p^+)$ corresponding to the two stable branches of the local inverse of W'.

Definition 4.2.1 (The function T). We define the continuous function

$$\mathrm{T} : [M^-, M^+] \setminus (p^-, p^+) \to [M^-, M^+] \setminus (p^-, p^+)$$

as follows: given $q \in [M^-, M^+] \setminus (p^-, p^+)$, $\mathrm{T}(q)$ is the unique point in $[M^-, M^+] \setminus (p^-, p^+)$ different from q, such that

$$W'(q) = W'(\mathrm{T}(q)).$$

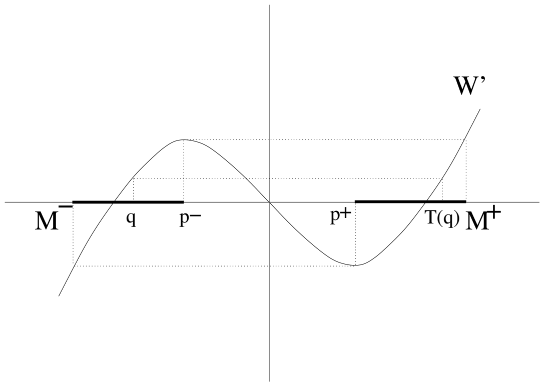

Figure 4.1: Graph of the function W'. In bold the set $[M^-, M^+] \setminus (p^-, p^+)$. The point $\mathrm{T}(q)$ for $q \in [M^-, M^+] \setminus (p^-, p^+)$.

The map T satisfies

$$\mathrm{T} \circ \mathrm{T} = \mathrm{id}.$$

Furthermore, it is strictly increasing, smooth on the interior of its domain,

$$q \in (p^+, M^+) \Rightarrow \mathrm{T}(q) \in (M^-, p^-), \qquad \mathrm{T}(p^+) = M^-,\ \mathrm{T}(M^+) = p^-,$$

$$q \in (M^-, p^-) \Rightarrow \mathrm{T}(q) \in (p^+, M^+), \qquad \mathrm{T}(M^-) = p^+,\ \mathrm{T}(p^-) = M^+,$$

and $\lim_{q \downarrow M^-} T'(q) = +\infty$, $\lim_{q \uparrow p^-} T'(q) = 0$.

Definition 4.2.2 (**Local unstable region of u**)**.** Given a Lipschitz function $u : \mathbb{T} \to \mathbb{R}$ and a point $x \in \mathbb{T}$ where u is differentiable, we write

$$x \in \Sigma_L(u)$$

if $u_x(x) \in (p^-, p^+)$. We call $\Sigma_L(u)$ the local unstable region of u.

4.3 Semidiscrete scheme

In what follows, given $h > 0$ sufficiently small, we assume $\mathbb{T}$ to be discretized with a grid of N_h subintervals of equal length h.
$PL_h(\mathbb{T})$ is the N_h-dimensional vector subspace of $\mathrm{Lip}(\mathbb{T})$ of all piecewise linear functions defined on the grid. $PC_h(\mathbb{T})$ is the N-dimensional vector subspace of $L^2(\mathbb{T})$ of all left-continuous piecewise constant functions on the grid.

Given $u \in PL_h(\mathbb{T})$ (resp. $u \in PC_h(\mathbb{T})$) we denote with $u_1, \dots, u_{N_h}$ the coordinates of u with respect to the basis of the hat (resp. flat) functions, and $u \in PL_h(\mathbb{T})$ will be identified with $(u_1, \dots, u_{N_h}) \in \mathbb{R}^{N_h}$, where $u_i := u(ih)$, $i = 1, \dots, N_h$, and $u_0 := u_{N_h}$. $PL_h(\mathbb{T})$ is endowed with the norm

$$\|u\|^2_{PL_h(\mathbb{T})} := h \sum_{i=1}^{N_h} (u_i)^2. \tag{4.11}$$

Notice that on $PL_h(\mathbb{T})$ the norms $\|\cdot\|_{PL_h(\mathbb{T})}$ and $\|\cdot\|_{L^2(\mathbb{T})}$ are equivalent, since a direct computation gives

$$\|u\|_{L^2(\mathbb{T})} \le \|u\|_{PL_h(\mathbb{T})} \le \sqrt{\frac{3}{2}}\, \|u\|_{L^2(\mathbb{T})}, \qquad u \in PL_h(\mathbb{T}).$$

Moreover these two norms coincide on $PC_h(\mathbb{T})$.
We define the linear maps $D_h^{\pm} : PL_h(\mathbb{T}) \to PC_h(\mathbb{T})$ as

$$(D_h^- u)_i = \frac{1}{h}(u_i - u_{i-1}), \quad (D_h^+ u)_i = \frac{1}{h}(u_{i+1} - u_i), \quad i \in \{1, \dots, N_h\}, \tag{4.12}$$

where $u_{N_h+1} := u_1$. Notice that $(D_h^- u)_{i+1} = (D_h^+ u)_i$. In addition, the following discrete integration by parts formula holds for functions $u, v \in PL_h(\mathbb{T})$ (hence with $u_0 = u_{N_h}$ and $v_0 = v_{N_h}$):

$$\sum_{i=1}^{N_h} (D_h^- u)_i v_i = -\sum_{i=1}^{N_h} u_i (D_h^+ v)_i. \tag{4.13}$$

It is clear that if $u \in PL_h(\mathbb{T})$ and $i \in \{1, \dots, N_h\}$, then $(D_h^- u)_i = u_x$ in the interval $((i-1)h, ih)$.

4.3.1 The semidiscrete scheme: the system of ODEs

The restriction F_h of F to $PL_h(\mathbb{T})$ is a smooth function of N variables and reads as

$$F_h(u) = h \sum_{i=1}^{N_h} W\left((D_h^- u)_i\right) = h \sum_{i=1}^{N_h} W\left(\frac{u_i - u_{i-1}}{h}\right), \qquad u \in PL_h(\mathbb{T}).$$

The $L^2(\mathbb{T})$-gradient flow of F_h (or equivalently the gradient flow of F_h with respect to the scalar product producing the norm $\|\cdot\|_{PL_h(\mathbb{T})}$ in (4.11)) on $PL_h(\mathbb{T})$ is expressed by

$$\begin{aligned} \frac{du_i}{dt} &= -\frac{1}{h}\frac{\partial F_h}{\partial u_i} = \frac{1}{h}\left\{ W'\left(\frac{u_{i+1} - u_i}{h}\right) - W'\left(\frac{u_i - u_{i-1}}{h}\right)\right\} \\ &= \left(D_h^+ W'(D_h^- u)\right)_i \end{aligned}$$

for any $i \in \{1, \ldots, N_h\}$, with the periodicity conditions $u_0 = u_{N_h}$, $u_1 = u_{N_h+1}$.
As already said in the introduction, we will be interested in the asymptotic limit, as $h \downarrow 0$, of space-periodic solutions u^h to the system of ordinary differential equations (4.6) with initial condition $\overline{u}^h$ (allowing initial conditions depending on h will be crucial).
The following two theorems taken from [60, Theorem 2.1 and Prop. 2.3] are the starting point of our analysis. Given a function $v : \mathbb{T} \times [0, +\infty) \to \mathbb{R}$ and $t \in [0, +\infty)$, we let $v(t)$ be the function on $\mathbb{T}$ defined as $v(t)(x) := v(x, t)$.

Theorem 4.3.1 (Properties of u^h). *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that*

$$\sup_h \|D_h^- \ \overline{u}^h\|_{L^\infty(\mathbb{T})} =: C < +\infty. \tag{4.14}$$

Then there exists a unique solution $u^h \in \mathcal{C}^\infty([0, +\infty); PL_h(\mathbb{T}))$ of (4.6) and

$$\|D_h^- u^h(t)\|_{L^\infty(\mathbb{T})} \leq \max\Big(C, M^+\Big), \qquad t \in [0, +\infty). \tag{4.15}$$

Moreover

$$\frac{d}{dt} F_h(u^h(t)) \ = - \left\| \frac{du^h}{dt} \right\|^2_{PL_h(\mathbb{T})} \leq \ 0, \qquad t \in (0, +\infty). \tag{4.16}$$

Note in particular[4] that
$u^h \in AC_2\left([0, +\infty); L^2(\mathbb{T})\right) \subset \mathcal{C}^{\frac{1}{2}}\left([0, +\infty); L^2(\mathbb{T})\right)$, and

$$\int_{\mathbb{T}} u^h(x, t)\, dx = \int_{\mathbb{T}} \overline{u}^h(x)\, dx, \qquad t \in [0, +\infty). \tag{4.17}$$

From (4.15) it follows that the Lipschitz constant of the solution u^h is conserved, provided it is larger[5] than M^+.

Remark 4.3.2 (**Uniform L^∞-bound**). Despite the maximum and minimum principles on u^h in general are not valid, from (4.17) and (4.15) it follows that, if (4.14) holds, then

$$\sup_h \|u^h\|_{L^\infty(\mathbb{T}\times(0,+\infty))} < +\infty. \tag{4.18}$$

[4] $AC_2([0, +\infty); L^2(\mathbb{T}))$ denotes the space of absolutely continuous functions from $[0, +\infty)$ to $L^2(\mathbb{T})$ with derivative in L^2, see [5].

[5] Small Lipschitz constants are not preserved, due to the formation of microstructures in correspondence of the concave region of W.

Theorem 4.3.3 (Preserving avoidance of $\Sigma_L(u^h)$). *If $\overline{u}^h \in PL_h(\mathbb{T})$ satisfies*

$$p^+ \leq \left|(D_h^- \overline{u}^h)_j\right| \leq M^+, \qquad j \in \{1, \ldots, N_h\}, \tag{4.19}$$

and if $i \in \{1, \ldots, N_h\}$, then

$$p^+ \leq (D_h^- \overline{u}^h)_i \leq M^+ \quad \Rightarrow \quad p^+ \leq (D_h^- u^h(t))_i \leq M^+, \quad t \in [0, +\infty),$$

$$M^- \leq (D_h^- \overline{u}^h)_i \leq p^- \quad \Rightarrow \quad M^- \leq (D_h^- u^h(t))_i \leq p^-, \quad t \in [0, +\infty). \tag{4.20}$$

In particular

$$p^+ \leq \left|(D_h^- u^h(t))_j\right| \leq M^+, \qquad j \in \{1, \ldots, N_h\},\ t \in [0, +\infty). \tag{4.21}$$

Therefore, if the initial data $\overline{u}^h$ have slopes which avoid[6] the concave region (p^-, p^+) of W then, as a consequence of (4.21), the same property is shared by the discrete solutions u^h, namely

$$\Sigma_L(\overline{u}^h) = \emptyset \quad \Rightarrow \quad \Sigma_L(u_h(t)) = \emptyset, \qquad t \geq 0. \tag{4.22}$$

We will make repeated use of this fact in the sequel (for instance in the proof of Theorem 4.6.7).

4.3.2 A constrained problem on slopes

In this section we formalize the idea of expressing a "macroscopic" gradient (denoted below by p) at a given point $x \in \mathbb{T}$ with a percentage σ of, say, negative "microscopic" gradient (indicated below by q, see also (4.26)) and a remaining percentage $1-\sigma$ of positive "microscopic" gradient (indicated below by $\mathrm{T}(q)$, where the map T is introduced in Definition 4.2.1). We stress that, by construction, q and $\mathrm{T}(q)$ are required not to lie in the concave region (p^-, p^+) of W. At the end of the procedure, the main concept will be the map $q(p, \sigma)$ in Definition 4.3.6, which will be crucial in order to "prepare" the initial datum $\overline{u}$ using an approximating sequence $(\overline{u}^h)$. We also anticipate here that in order to prepare $\overline{u}$, we will need to constrain the values of p (see (4.25)), and this will be source of a restriction on our initial datum.

Let $p \in \mathbb{R}$ and $\sigma \in [0, 1]$ be given. Let us consider the following problem: find q such that

$$q \in [M^-, M^+] \setminus (p^-, p^+) \tag{4.23}$$

[6] The functions $\overline{u}^h$ will be approximations of $\overline{u}$ which, on the contrary, has in general a *nonempty* local unstable region.

and

$$\sigma q + (1-\sigma)\mathrm{T}(q) = p. \tag{4.24}$$

It is not difficult to prove that if (4.24) admits a solution q, then the set $\{q, \mathrm{T}(q)\}$ is unique; namely, if q_1 and q_2 are two solutions of (4.24), i.e.,

$$p = \sigma q_1 + (1-\sigma)\mathrm{T}(q_1) = \sigma q_2 + (1-\sigma)\mathrm{T}(q_2),$$

then either $q_2 = q_1$ or $q_2 = \mathrm{T}(q_1)$. Indeed, if $\sigma \in \{0, 1\}$ this assertion is obvious. Assume then $\sigma \in (0, 1)$. Since q_1 and $\mathrm{T}(q_1)$ belong to different connected components of $[M^-, M^+] \setminus (p^-, p^+)$ (and the same holds for q_2 and $\mathrm{T}(q_2)$), up to exchanging q_1 with $\mathrm{T}(q_1)$ and q_2 with $\mathrm{T}(q_2)$ we can assume that $q_1, q_2 \in [p^+, M^+]$. Assume by contradiction that $q_1 \neq q_2$. Supposing without loss of generality that $q_1 < q_2$ we get, recalling that T is increasing,

$$0 > q_1 - q_2 = \frac{1-\sigma}{\sigma}(\mathrm{T}(q_2) - \mathrm{T}(q_1)) > 0,$$

which is absurd.

We now come to the existence of a solution to equations (4.23) and (4.24), noticing that they are not always solvable: for instance, if $\sigma = 1$, then $q = p$ which implies that the problem does not have a solution unless $p \in [M^-, M^+] \setminus (p^-, p^+)$. Similarly, if $\sigma = 0$, then $\mathrm{T}(q) = p$ which again implies that the problem has not a solution, unless $p \in [M^-, M^+] \setminus (p^-, p^+)$.

Example 4.3.4. Let $\sigma = 1/2$. Then equation (4.24) gives, for q as in (4.23),

$$\frac{1}{2\sqrt{3}} = \frac{p^- + M^+}{2} \geq \frac{q + \mathrm{T}(q)}{2} = p \geq \frac{p^+ + M^-}{2} = -\frac{1}{2\sqrt{3}},$$

which constraints p to $|p| \leq \frac{1}{2\sqrt{3}}$.

Definition 4.3.5 (**The set-valued map G**). Given $\sigma \in [0, 1]$, we set

$$G(\sigma) := [\sigma M^- + (1-\sigma)p^+, \sigma p^- + (1-\sigma)M^+].$$

For $q \in [p^-, M^-]$, as a consequence of the fact that the function

$$q \in [p^-, M^-] \to \sigma q + (1-\sigma)\mathrm{T}(q)$$

is increasing, we deduce (generalizing Example 4.3.4) that the inclusion

$$p \in G(\sigma) \tag{4.25}$$

implies that (4.23), (4.24) admit a unique solution $\{q, \mathrm{T}(q)\}$.

We are now in a position to give the following definition[7].

Definition 4.3.6 (The function q). Let $p \in G(\sigma)$ and let $\{q, \mathrm{T}(q)\}$ be the solution to (4.23) and (4.24). We set

$$q(p,\sigma) := \min\{q, \mathrm{T}(q)\} < 0. \tag{4.26}$$

Observe that $q(p,1) = p$ and $q(p,0) = \mathrm{T}(p)$.

Remark 4.3.7 (**Regularity of the function $q(\cdot,\sigma)$**). For all $\sigma \in [0,1]$, the function $q(\cdot,\sigma) : G(\sigma) \to [M^-, p^-]$ satisfies

$$q(\cdot,\sigma) \in \mathcal{C}^\infty(\mathrm{int}(G(\sigma))) \cap \mathcal{C}^0(G(\sigma))$$

and

$$q_p(\cdot,\sigma) > 0 \qquad \text{in } \mathrm{int}(G(\sigma)),$$

where $\mathrm{int}(G(\sigma)) = (\sigma M^- + (1-\sigma)p^+, \sigma p^- + (1-\sigma)M^+)$. In addition, differentiating (4.24) with respect to p and taking $\sigma \in (0,1)$ yields $q_p(p,\sigma) = \frac{1}{\sigma+(1-\sigma)\mathrm{T}'(q(p,\sigma))}$, and

$$\text{for } \sigma \neq 1 \qquad \lim_{p \downarrow \min G(\sigma)} q_p(p,\sigma) = 0,$$

$$\text{for } \sigma \neq 0 \qquad \lim_{p \uparrow \max G(\sigma)} q_p(p,\sigma) = \frac{1}{\sigma},$$

and $\lim_{p \uparrow \max G(1)} q_p(p,1) = +\infty$.

4.4 Admissible initial data

The instability of (4.2) implies that the limit of (u^h) as $h \to 0^+$ could depend on the sequence $(\overline{u}^h)$ chosen in (4.6) for approximating $\overline{u}$ in $L^\infty(\mathbb{T})$. In this section we specify which kind of sequences of initial data we will consider. Let us start with the following definition, which fixes our compatible initial data $\overline{u}$. $\mathbb{Q} \subset \mathbb{R}$ denotes the set of rational numbers.

Definition 4.4.1 (The class $\mathcal{D}$ and the percentage $\overline{\varrho}$). We say that a function $\overline{u}$ belong to the class $\mathcal{D}$ if the following properties hold:

(i) $\overline{u} \in \mathrm{Lip}(\mathbb{T})$;

[7]Our results still hold with obvious modifications if in place of min we take max in (4.26). For instance, substitute the sentence prc-percentage of negative slopes with prc-percentage of positive slopes in Definition 4.4.1.

(ii) there exists a function $\overline{\varrho} \in L^\infty(\mathbb{T};[0,1])$ such that $\overline{\varrho}(\mathbb{T})$ is a finite subset of $\mathbb{Q}$ and $\overline{\varrho}^{-1}(\alpha)$ is a (nontrivial) subinterval of $\mathbb{T}$ for all $\alpha \in \overline{\varrho}(\mathbb{T})$, and

$$\overline{u}_x(x) \in G(\overline{\varrho}(x)), \qquad \text{a.e. } x \in \mathbb{T}. \tag{4.27}$$

The function $\overline{\varrho}$ will be called a piecewise constant rational percentage (pcr-percentage for short) of negative slopes for $\overline{u}$.

Notice that from (4.27) it follows that $\mathrm{lip}(\overline{u}) \leq M^+$.

Remark 4.4.2. From our definitions it follows that, given $\overline{u} \in \mathcal{D}$ and a pcr-percentage $\overline{\varrho}$ of negative slopes for $\overline{u}$, for almost every $x \in \mathbb{T}$ we have

$$\begin{aligned} q(\overline{u}_x(x), \overline{\varrho}(x)) < 0, \\ \overline{\varrho}(x) q\left(\overline{u}_x(x), \overline{\varrho}(x)\right) + (1-\overline{\varrho}(x)) \mathrm{T}\left(q(\overline{u}_x(x), \overline{\varrho}(x))\right) = \overline{u}_x(x). \end{aligned} \tag{4.28}$$

Example 4.4.3. Let $\overline{u} \in \mathrm{Lip}(\mathbb{T})$ satisfy the following property: there exists a finite set of points of $\mathbb{T}$ such that if I is a connected component of the complement of this set, then $\overline{u}_{|\overline{I}} \in \mathcal{C}^1(\overline{I})$, and

$$\text{either } \overline{u}_x(I) \subset (p^-, p^+) \text{ or } \overline{u}_x(I) \subset [M^-, p^-] \text{ or } \overline{u}_x(I) \subset [p^+, M^+]. \tag{4.29}$$

Then $\overline{u} \in \mathcal{D}$. Indeed, it is enough to choose

$$\overline{\varrho} : \mathbb{T} \to \left\{0, \frac{1}{2}, 1\right\}$$

as follows: if $x \in \mathbb{T}$ is a differentiability point of u,

$$\overline{\varrho}(x) := \begin{cases} 1 & \text{if } \overline{u}_x(x) \subset [M^-, p^-], \\ \frac{1}{2} & \text{if } \overline{u}_x(x) \subset (p^-, p^+), \\ 0 & \text{if } \overline{u}_x(x) \subset [p^+, M^+]. \end{cases}$$

Then, remembering that $G(0) = [p^+, M^+]$, $G(\frac{1}{2}) = \frac{1}{2}[M^- + p^+, p^- + M^+]$, and $G(1) = [M^-, p^-]$, it is immediate to see that, in view of (4.29), condition (4.27) is satisfied. Note that, in this case,

$$\mathrm{lip}\Big(\overline{u}_{|\Sigma_L(\overline{u})}\Big) \leq \frac{p^- + M^+}{2}.$$

Remark 4.4.4. The condition in (ii) on the structure of a prc-percentage $\overline{\varrho}$ is a technical assumption used in the proof of Theorem 4.6.9. Such an assumption will be removed in Theorem 4.6.14. We recall also that

condition (4.27) is needed to construct the function $q(p,\sigma)$ illustrated in Section 4.3.2, applied with the choice $p = \overline{u}(x)$ and $\sigma = \overline{\varrho}(x)$.
Notice that

$$\{v \in \mathcal{C}^1(\mathbb{T}) : \mathrm{lip}(v) \leq M^+\} \subset \mathcal{D} \subset \{v \in \mathrm{Lip}(\mathbb{T}) : \mathrm{lip}(v) \leq M^+\},$$

with dense inclusions with respect to the $L^\infty(\mathbb{T})$-norm.

4.4.1 Admissible sequences of initial data

We now specify the approximating sequences $(\overline{u}^h)$ that we will be consider in the semidiscrete scheme (4.6).

Definition 4.4.5 (Admissible sequences). Let $\overline{u} \in \mathcal{D}$. We say that the sequence $(\overline{u}^h)$ is admissible for $\overline{u}$ if the following properties hold:

- $\overline{u}^h \in PL_h(\mathbb{T})$;
- $\mathrm{lip}(\overline{u}^h) \leq M^+$;
- $\Sigma_L(\overline{u}^h) = \emptyset$;
- $\lim_{h \to 0^+} \|\overline{u}^h - \overline{u}\|_{L^\infty(\mathbb{T})} = 0$.

Observe that if $(\overline{u}^h)$ is admissible for $\overline{u}$, then

$$\sup_h \Big(\|\overline{u}^h\|_{L^\infty(\mathbb{T})} + F_h(\overline{u}^h) \Big) < +\infty. \tag{4.30}$$

Given $\overline{u} \in \mathcal{D}$, we now construct an admissible sequence $(\overline{u}^h)$ for $\overline{u}$. The idea for constructing $(\overline{u}^h)$ is to keep the values of $\overline{u}$ out of $\Sigma_L(\overline{u})$, and to take a suitable approximation of $\overline{u}$ in $\Sigma_L(\overline{u})$, in particular ensuring that the avoidance condition $\Sigma_L(\overline{u}^h) = \emptyset$ is satisfied. To do that, we will make use of the existence of a pcr-percentage of negative slopes for $\overline{u}$.
Let us first introduce the following "averaged discrete gradient", where the average is taken over a scale of order nh.

Definition 4.4.6 (Averaged discrete gradients). Let $u \in PL_h(\mathbb{T})$ and $n \in \mathbb{N}$. We set

$$(D^-_{nh} u)_i := \frac{1}{n} \sum_{j=i-n+1}^{i} (D^-_h u)_j = \frac{u_i - u_{i-n}}{nh}, \qquad i \in \{1, \dots, N_h\}. \tag{4.31}$$

For example, if $n = 2$, equation (4.31) is nothing else than the discrete derivative in (4.12) (left), on a grid of size $2h$.
Of course, Definition 4.4.6 makes sense also for $n = 1$; however, the general case $n \in \mathbb{N}$ will be necessary in the proof of Theorem 4.6.7.

4.4.2 Construction of an admissible sequence for $\overline{u} \in \mathcal{D}$.

Let $\overline{u} \in \mathcal{D}$ and let $\overline{\varrho}$ be a pcr-percentage of negative slopes for $\overline{u}$. We have a set $\{\xi_1, \dots, \xi_M\} \subset \mathbb{T}$ of points with $\xi_j < \xi_{j+1}$ so that

$$\overline{\varrho}(x) = \sigma_j = \frac{m_j}{n_j} \in \mathbb{Q}, \qquad x \in (\xi_j, \xi_{j+1}),\ j \in \{1, \dots, M\},$$

where we let $\xi_{M+1} := \xi_1$.
We want to define $\overline{u}^h$ in $[\xi_j, \xi_{j+1}]$: we partition $[\xi_j, \xi_{j+1}]$ into subintervals of length $n_j h$ indicized by the index k (if the partition does not coincide exactly with $[\xi_j, \xi_{j+1}]$ then there may remain at most two small intervals adjacent to ξ_j and ξ_{j+1}, where we will set $\overline{u}^h := \overline{u}$). In each of these intervals of length $n_j h$, say $[kn_j h, (k+1)n_j h]$, the averaged slope of $\overline{u}$ is given by

$$\overline{p}_k := (D^-_{n_j h} \overline{u})_{(k+1)n_j}.$$

We want that:

- $\overline{u}^h$ has the same averaged slope $\overline{p}_k$ of $\overline{u}$ in $[kn_j h, (k+1)n_j h]$,
- $D^-_h \overline{u}^h$ belongs to the region where W is convex, namely it avoids the concave region of W,
- $\overline{u}^h$ is decreasing with slope $\overline{q}_k$ on the first m_j subintervals of $[kn_j h, (k+1)n_j h]$, and increasing with slope $\mathrm{T}(\overline{q}_k)$ on the remaining $n_j - m_j$ subintervals, where

$$\overline{q}_k := q\,(\overline{p}_k, \sigma_j) < 0.$$

Being the avergaed slope of $\overline{u}^h$ equal to $\overline{p}_k$, it follows that $D^-_h \overline{u}^h$ must be equal to $\overline{q}_k$ on m_j subintervals, and $\mathrm{T}(\overline{q}_k)$ on the remaining subintervals. Therefore we set $D^-_h \overline{u}^h = \overline{q}_k$ on the first m_j subintervals, and $D^-_h \overline{u}^h = \mathrm{T}(\overline{q}_k)$ on the remaining $n_j - m_j$ subintervals.
The definition is the following:

Definition 4.4.7 (Construction of $(\overline{u}^h)$). Let $j \in \{1, \dots, M\}$ and let $x \in [\xi_j, \xi_{j+1}]$ be a grid point of $\mathbb{T}$. We write $x = (kn_j + m)h$ for some $k, m \in \mathbb{N}$ with $m < n_j$. We define $\overline{u}^h(x)$ as follows:

– if $\{kn_jh, (k+1)n_jh\} \not\subset [\xi_j, \xi_{j+1}]$, we set
$$\overline{u}^h(x) := \overline{u}(x);$$

– if $\{kn_jh, (k+1)n_jh\} \subset [\xi_j, \xi_{j+1}]$ and $m \leq m_j$, we set
$$\overline{u}^h(x) := \overline{u}(kn_jh) + m\overline{q}_k h\,, \tag{4.32}$$

– if $\{kn_jh, (k+1)n_jh\} \subset [\xi_j, \xi_{j+1}]$ and $m > m_j$, we set
$$\overline{u}^h(x) := \overline{u}(kn_jh) + m_j\overline{q}_k h - (m-m_j)\mathrm{T}(\overline{q}_k)h\,. \tag{4.33}$$

The function $\overline{u}^h$ is then extended piecewise linearly out of the nodes of the grid.

Notice the signs on the right hand sides of (4.32) and (4.33): remember that $\overline{q}_k$ is negative by construction, while $\mathrm{T}(\overline{q}_k)$ is positive, hence $\overline{u}^h$ is decreasing up to m_j, and then increases.
By construction, we have the following result.

Lemma 4.4.8. *Let $\overline{u} \in \mathcal{D}$. Then the sequence $(\overline{u}^h)$ of Definition 4.4.7 is admissible for $\overline{u}$.*

4.5 A priori estimates

In order to pass to the limit in equation (4.6), we need to find several a priori estimates on approximate solutions u^h, under restriction (4.19) on the inital datum $\overline{u}^h$.

Proposition 4.5.1. *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that condition* (4.19) *holds. Let u^h be the corresponding solution given by Theorem 4.3.1. Then*
$$\frac{d}{dt}\left(\left\|\frac{d}{dt}u^h(t)\right\|_{PL_h(\mathbb{T})}\right) \leq 0, \tag{4.34}$$
and
$$\frac{d}{dt}\left(\left\|\frac{d}{dt}u^h(t)\right\|_{L^\infty(\mathbb{T})}\right) \leq 0. \tag{4.35}$$

Proof. Set for notational simplicity $w := \dfrac{du^h}{dt}$. We have, for $i \in \{1, \dots, N_h\}$ and using (4.6),
$$\begin{aligned}\frac{dw_i}{dt} =&\frac{d}{dt}\Big(D_h^+ W'\left((D_h^- u^h)_i\right)\Big) = D^+\frac{d}{dt}\Big(W'\left((D_h^- u^h)_i\right)\Big)\\ =&D_h^+\Big(W''\left((D_h^- u^h)_i\right) D_h^- w_i\Big).\end{aligned} \tag{4.36}$$

From inequalities (4.21) it follows

$$W''\left((D_h^- u^h(t))_i\right) \geq 0, \qquad t \in (0, +\infty). \tag{4.37}$$

Remembering the definition (4.11) of the $\|\cdot\|_{PL_h(\mathbb{T})}$ norm, and using the discrete integration by parts formula (4.13), we have

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}\left\|\frac{d}{dt}u^h(t)\right\|^2_{PL_h(\mathbb{T})}\right) =& \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N_h}(w_i)^2\right) \\
=& h\sum_{i=1}^{N_h} w_i D_h^+\Big(W''\left((D_h^- u)_i\right)(D_h^- w)_i\Big) \\
=& -h\sum_{i=1}^{N_h}(D_h^- w)_i W''\left((D_h^- u)_i\right)(D_h^- w)_i \leq 0,
\end{aligned}$$

which proves (4.34).
Let us now prove (4.35). Let $ih \in \mathbb{T}$ be a grid point where W takes its maximum. Then

$$(D_h^- w)_i \geq 0, \qquad (D_h^- w)_{i+1} \leq 0.$$

Recalling (4.37) we deduce

$$W''\left((D_h^- u^h)_i\right)(D_h^- w)_i \geq 0, \qquad W''\left((D_h^- u^h)_{i+1}\right)(D_h^- w)_{i+1} \leq 0.$$

From the above inequalities and (4.36) we obtain

$$\frac{dw_i}{dt} = D_h^+\Big(W''\left((D_h^- u^h)_i\right)(D_h^- w)_i\Big) \leq 0.$$

A similar proof holds for a minimum point. Then (4.35) follows. □

Corollary 4.5.2 (Convexity of $F_h(u^h)$). *The function $t \in [0, +\infty) \to F_h(u^h(t))$ is convex and nonincreasing.*

Proof. It is a consequence of (4.16) and (4.34). □

The next result shows that $W'(D_h^- u^h)$ are 1/2-Hölder continuous in space. This fact will allow to use the Ascoli-Arzelà theorem for passing to the limit in the nonlinear term $W'(D_h^- u^h)$ in (4.6).

Proposition 4.5.3 (Hölder continuity of $W'(D_h^- u^h)$). *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that (4.19) holds. Let u^h be the corresponding solution given by Theorem 4.3.1. Then for all $k, l \in \{1, \dots, N_h\}$ and $t \in (0, +\infty)$ we have*

$$\left|W'\left((D_h^- u^h(t))_l\right) - W'\left((D_h^- u^h(t))_k\right)\right| \leq \sqrt{|l-k|h}\,\left\|\frac{du^h}{dt}(t)\right\|_{PL_h(\mathbb{T})}. \tag{4.38}$$

Proof. Without loss of generality we assume $l > k$, and we write for notational simplicity
$$u = u^h.$$
We have, using the equality $(D_h^+ u)_i = (D_h^- u)_{i+1}$,

$$\begin{aligned}
&\Big| W'\Big((D_h^- u(t))_l\Big) - W'\Big((D_h^- u(t))_k\Big)\Big| \\
&= \Big| W'\Big((D_h^- u(t))_l\Big) - W'\Big((D_h^- u(t))_{l-1}\Big) + \ \dots \\
&\quad + W'\Big((D_h^- u(t))_{k+1}\Big) - W'\Big((D_h^- u(t))_k\Big)\Big| \\
&\leq \sum_{i=k}^{l-1} \Big| W'\Big((D_h^- u(t))_{i+1}\Big) - W'\Big((D_h^- u(t))_i\Big)\Big| \\
&\leq \sum_{i=k}^{l-1} \Big| W'\Big((D_h^+ u(t))_i\Big) - W'\Big((D_h^- u(t))_i\Big)\Big| ,
\end{aligned}$$

where we stress that any difference appearing in the last line involves the two operators D_h^- and D_h^+. Then, using Hölder's inequality and the definition (4.11) of $PL_h(\mathbb{T})$-norm, it follows

$$\begin{aligned}
&\Big| W'\Big((D_h^- u(t))_l\Big) - W'\Big((D_h^- u(t))_k\Big)\Big| \\
&\leq \sqrt{(l-k)h} \ \left(h \sum_{i=k}^{l-1} \left| \frac{1}{h}\Big(W'\left((D_h^+ u(t))_i\right) - W'\left((D_h^- u(t))_i\right)\Big)\right|^2 \right)^{1/2} \\
&= \sqrt{(l-k)h} \ \left(h \sum_{i=k}^{l-1} \left| D_h^+ W'\left((D_h^- u(t))_i\right)\right|^2 \right)^{1/2} \\
&\leq \sqrt{(l-k)h} \ \left(h \sum_{i=k}^{l-1} \left| \frac{du_i}{dt}(t)\right|^2 \right)^{1/2} = \sqrt{(l-k)h} \ \left\| \frac{du}{dt}(t)\right\|_{PL_h(\mathbb{T})} ,
\end{aligned}$$

which gives the desired result. □

The Hölder space constant of $W'(D_h^- u^h)$ is uniform with respect to time, since the quantity $\left\| \frac{du^h}{dt}\right\|_{PL_h(\mathbb{T})}$ on the right hand side of (4.38) is nonincreasing in time by inequality (4.34). More precisely, the following result holds.

Corollary 4.5.4. *Let $\overline{u}^h \in PL_h(\mathbb{T})$ and suppose that* (4.19) *holds. Let u^h be the corresponding solution given by Theorem 4.3.1 and let*

$\tau > 0$. *Then for all* $k, l \in \{1, \dots, N_h\}$ *and* $t > \tau$ *we have*

$$\left| W'\left((D_h^- u^h(t))_l\right) - W'\left((D_h^- u^h(t))_k\right) \right| \leq \sqrt{|l-k|h} \sqrt{\frac{F_h(\overline{u}^h)}{\tau}}. \tag{4.39}$$

Proof. From estimate (4.38) and using formula (4.16) we have

$$\left| W'\left((D_h^- u^h(t))_k\right) - W'\left((D_h^- u^h(t))_l\right) \right| \leq \sqrt{|l-k|h} \left| \frac{d}{dt} F_h(u^h(t)) \right|. \tag{4.40}$$

If $f : [0, +\infty) \to (0, +\infty)$ is a convex nonincreasing differentiable function and $\tau > 0$, we have $f'(\tau) \geq \frac{f(\tau)-f(0)}{\tau} \geq -\frac{f(0)}{\tau}$, hence $\dfrac{f(0)}{\tau} \geq |f'(\tau)|$. Then (4.39) follows from Corollary 4.5.2 and the latter inequality. □

4.6 Main results and the limit problem

Set

$$\mathcal{X} := AC_2\Big([0, +\infty); L^2(\mathbb{T})\Big) \cap L^\infty\Big((0, +\infty); \mathrm{Lip}_{M^+}(\mathbb{T})\Big) \cap L^\infty\Big(\mathbb{T} \times (0, +\infty)\Big), \tag{4.41}$$

where $\mathrm{Lip}_{M^+}(\mathbb{T}) := \{u \in \mathrm{Lip}(\mathbb{T}) : \mathrm{lip}(u) \leq M^+\}$ is endowed with the topology of uniform convergence.

Remark 4.6.1. Let $(\overline{u}^h)$ satisfy (4.14) and assume in addition that (4.30) holds. Let u^h be the solution to (4.6). Then

$$u^h \in \mathcal{X}. \tag{4.42}$$

A first compactness result on the sequence (u^h) is given by the following proposition.

Proposition 4.6.2 (Weak compactness). *Let* $(\overline{u}^h)$ *satisfy* (4.14) *and* (4.30)*. Let* u^h *be the solution to* (4.6)*. Then there exist a function*

$$u \in \mathcal{X} \quad \text{with} \quad u(0) = \overline{u},$$

and a (not relabelled) subsequence (u^h) *such that*

(i) $\lim_{h \to 0^+} u^h = u$ *weakly in* $H^1_{\mathrm{loc}}([0, +\infty); L^2(\mathbb{T}))$,

(ii) $\lim_{h \to 0^+} u^h = u$ *weakly* in* $L^\infty_{\mathrm{loc}}((0, +\infty); \mathrm{Lip}_{M^+}(\mathbb{T}))$,

and

$$\int_{\mathbb{T}} u(x,t)\ dx = \int_{\mathbb{T}} \overline{u}(x)\ dx, \qquad t \geq 0. \tag{4.43}$$

Moreover there exists a constant $C_1 > 0$ such that

$$|u_x| \leq C_1 \qquad \text{in } \mathbb{T} \times [0, +\infty).$$

Proof. From (4.16) and (4.30) we have that, for any $T > 0$, the sequence (u^h) is bounded in $H^1([0,T]; L^2(\mathbb{T}))$. Therefore, passing to a (not relabelled) subsequence, we have conclusion (i) for a function $u \in AC_2([0,+\infty); L^2(\mathbb{T}))$. Using (4.15) we have assertion (ii) and the inclusion $u \in L^\infty\left((0,+\infty); \mathrm{Lip}_{M^+}(\mathbb{T})\right)$. In particular $u^h(\cdot,t) \to u(\cdot,t)$ uniformly for any $t \geq 0$, which implies $u(\cdot,0) = \lim_{h\to 0^+} \overline{u}^h(\cdot) = \overline{u}(\cdot)$. The inclusion $u \in L^\infty(\mathbb{T} \times (0,+\infty))$ follows from (4.18), hence $u \in \mathcal{X}$. Eventually, equality (4.43) follows from (4.17). $\square$

Before observing another compactness property of the sequence (u^h), we need the following result, which is independent of the evolution equation (4.2), being a property of the function space $\mathcal{X}$ only.

Proposition 4.6.3. *Let $v \in \mathcal{X}$. Then there exists a constant $C_2 > 0$ such that*

$$|v(x,t) - v(y,s)| \leq C_2 \left(|x-y| + |t-s|^{1/3}\right),\ x, y \in \mathbb{T},\ s, t \in (0,+\infty). \tag{4.44}$$

Proof. Let $I \subseteq \mathbb{T}$ be an interval of length $|I|$. Fix $x, y \in I$ and $s, t \in (0,\infty)$. Since $v \in L^\infty\left((0,+\infty); \mathrm{Lip}_{M^+}(\mathbb{T})\right)$, it follows that $v(\cdot,t)$ is Lipschitz continuous uniformly with respect to t, therefore

$$\begin{aligned}
&|v(x,t) - v(y,s)| = \frac{1}{|I|}\int_I |v(x,t) - v(y,s)|\ dz \\
&\leq \frac{1}{|I|}\int_I \big(|v(x,t) - v(z,t)| + |v(z,t) - v(z,s)| + |v(z,s) - v(y,s)|\big)\ dz \\
&\leq \frac{1}{|I|}\int_I |v(z,t) - v(z,s)|\ dz + C|I|,
\end{aligned}$$

where $C > 0$ is a suitable constant. Hence, using twice Hölder's inequality and the assumption $v \in AC_2([0,+\infty); L^2(\mathbb{T}))$ we deduce

$$\begin{aligned}
|v(x,t)-v(y,s)| &\leq \frac{1}{|I|^{1/2}} \left(\int_{\mathbb{T}} |v(z,t)-v(z,s)|^2 \, dz \right)^{1/2} + C|I| \\
&\leq \frac{\sqrt{|t-s|}}{|I|^{1/2}} \left(\int_{\mathbb{T}\times[0,+\infty)} (v_t)^2 \, dzdt \right)^{1/2} + C|I| \\
&\leq \kappa \left(\frac{\sqrt{|t-s|}}{|I|^{1/2}} + |I| \right),
\end{aligned}$$

where $\kappa > 0$ is a suitable constant.
Since $|x-y| \leq |I| \leq 1$, we have

$$\begin{aligned}
|v(x,t)-v(y,s)| &\leq \kappa \min_{\lambda\in[|x-y|,1]} \left(\frac{\sqrt{|t-s|}}{\lambda^{1/2}} + \lambda \right) \\
&\leq C_2 \left(|t-s|^{1/3} + |x-y| \right)
\end{aligned} \tag{4.45}$$

for a constant $C_2 > 0$. □

Corollary 4.6.4 (Hölder continuity and uniform convergence of (u^h)). *] Let $(\overline{u}^h)$, (u^h) and u be as in Proposition 4.6.2. Then there exists a constant $C_3 > 0$ such that*

$$|u^h(x,t) - u^h(y,s)| \leq C_3 \left(|x-y| + |t-s|^{1/3} \right), \quad \begin{array}{l} x, y \in \mathbb{T}, \\ s,t \in (0,+\infty), \\ h \in \mathbb{N}. \end{array} \tag{4.46}$$

Moreover there exists a (not relabelled) subsequence (u^h) such that

$$u^h \to u \text{ uniformly on compact subsets of } \mathbb{T}\times[0,+\infty). \tag{4.47}$$

Proof. The proof of (4.46) is the same as the proof of (4.44), recalling (4.15) and (4.16). The uniform convergence of (u^h) then follows from (4.18), (4.46), and the Ascoli-Arzelà theorem. □

4.6.1 Compactness of gradients

In order to pass to the limit in problem (4.6) we need some compactness properties on the discrete gradients of the sequence of solutions to (4.6). We cannot hope to have simply a compactness of the sequence $(D_h^- u^h)$ of "microscopic" gradients, because of their oscillations (microstructures) in the local unstable region. Therefore, we will take a

suitable average of these gradients giving raise to a sort of "macroscopic" gradient $D^-_{nh}u^h$, $n \in \mathbb{N}$. We will gain a uniform Hölder's estimate for the sequence $\left(W'\left(D^-_h u^h\right)\right)$.
We start with the following result, which shows that, for functions in $\mathcal{X}$, Hölder continuity in time can be obtained from Hölder continuity in space.

Lemma 4.6.5. *Let $v \in \mathcal{X}$. Suppose that there exist $C > 0$ and $\alpha \in (0,1)$ such that*

$$|v_x(x,t) - v_x(y,t)| \leq C|x-y|^\alpha, \qquad x, y \in \mathbb{T},\ t > 0. \tag{4.48}$$

Then there exists a constant $C_4 > 0$ such that

$$|v_x(x,t) - v_x(x,s)| \leq C_4|t-s|^{\frac{\alpha}{3(\alpha+1)}}, \qquad x \in \mathbb{T},\ s,t > 0.$$

Proof. For $\delta > 0$ we have

$$\begin{aligned}
v(x+\delta,t) - v(x,t) &= \int_x^{x+\delta} v_x(y,t)\, dy \\
&= v_x(x,t)\,\delta + \int_x^{x+\delta} (v_x(y,t) - v_x(x,t))\, dy, \\
v(x+\delta,s) - v(x,s) &= \int_x^{x+\delta} v_x(y,s)\, dy \\
&= v_x(x,s)\,\delta + \int_x^{x+\delta} (v_x(y,s) - v_x(x,s))\, dy.
\end{aligned}$$

Subtracting the two equations we have

$$\begin{aligned}
v_x(x,t) - v_x(x,s) &= \frac{1}{\delta}\Big(v(x+\delta,t) - v(x+\delta,s) + v(x,s) - v(x,t)\Big) \\
&\quad + \frac{1}{\delta}\int_x^{x+\delta} \Big[(v_x(x,t) - v_x(y,t)) + (v_x(y,s) - v_x(x,s))\Big]\, dy \\
&=: \mathrm{I} + \mathrm{II}.
\end{aligned}$$

Recalling (4.44) we have

$$\mathrm{I} \leq \frac{2C_2}{\delta}\ |t-s|^{1/3}.$$

Moreover, from hypothesis (4.48) it follows

$$\mathrm{II} \leq 2C\delta^\alpha.$$

Therefore, for $\kappa := 2\max(C_2, C)$ we have

$$|v_x(x,t) - v_x(x,s)| \le \kappa \left(\frac{|t-s|^{\frac{1}{3}}}{\delta} + \delta^\alpha \right). \tag{4.49}$$

The thesis follows by minimizing the right hand side of (4.49) among all $\delta > 0$. □

Lemma 4.6.5 has a the following discrete version, the proof of which is omitted being similar to the previous proof.

Lemma 4.6.6. *Let $v \in \mathcal{X}$. Suppose that there exist $C > 0$ and $\alpha \in (0,1)$ such that*

$$|D_h^- v(hi,t) - D_h^- v(hj,t)| \le C\,(h|i-j|)^\alpha, \quad i,j \in \{1,\dots,N_h\},\ t>0.$$

Then there exists a constant $C_5 > 0$ such that

$$|D_h^- v(hi,t) - D_h^- v(hj,s)| \le C_5|t-s|^{\frac{\alpha}{3(\alpha+1)}}, \quad i,j \in \{1,\dots,N_h\},\ s,t>0.$$

We are now in a position to prove the following compactess result for the averaged discrete gradients.

Theorem 4.6.7 (Compactness of discrete gradients). *Let $\overline{u} \in \mathcal{D}$ and let $\overline{\varrho}$ be a pcr-percentage of negative slopes for $\overline{u}$. Accordingly, let $M \in \mathbb{N}$ and $\{\xi_1,\dots,\xi_M\} \subset \mathbb{T}$ be so that $\overline{\varrho}$ takes a constant rational value σ_i on each (ξ_i, ξ_{i+1}) and (4.27) holds. Finally, let $(\overline{u}^h)$ be the admissible sequence for $\overline{u}$ constructed in Definition 4.4.7 and let u^h be the solution of (4.6). Then there exist a function*

$$u \in \mathcal{X} \quad \text{with} \quad u(0) = \overline{u},$$

and a (not relabelled) subsequence (u^h) converging to u uniformly on compact subsets of $\mathbb{T} \times [0,+\infty)$, such that

$$\lim_{h\to 0^+} D^-_{n_i h} u^h = u_x \tag{4.50}$$

uniformly on compact subsets of $(\xi_i, \xi_{i+1}) \times (0,+\infty)$, for all $i \in \{1,\dots,M\}$.

Proof. Let u and (u^h) be the function and the subsequence given by Proposition 4.6.2 respectively. Given $\tau > 0$, from (4.39) we have for any $l,k \in \{1,\dots,N_h\}$ and $t > \tau$,

$$\left|W'\left((D_h^- u^h(t))_l\right) - W'\left((D_h^- u^h(t))_k\right)\right| \le \sqrt{\frac{F_h(\overline{u}^h)}{\tau}}\sqrt{h|l-k|}. \tag{4.51}$$

We would like to deduce from (4.51) an estimate for $|(D_h^- u^h(t))_l - (D_h^- u^h(t))_k|$, by locally "inverting"[8] W' on the interval $(W'(p^+), W'(p^-))$. In general, such an estimate is not possible; however, we are able to prove an Hölder estimate for the averaged discrete gradients (see (4.58) below), and it is precisely here that $\overline{\varrho}$ takes a role: actually, estimate (4.58) is the motivation for introducing the notion of averaged gradient in Definition 4.4.6.
Remember that our construction of $(\overline{u}^h)$ is made so that (4.19) holds (see the third condition in Definition 4.4.5), and therefore (4.21) ensures that the discrete gradient of u^h avoids the concave region of W, i.e.,

$$(D_h^- u^h(t))_l \notin (p^-, p^+), \qquad l \in \{1, \ldots, N_h\}, \qquad t \in [0, +\infty).$$

From (4.51) we deduce that there exists a constant $C_1(\tau) > 0$ such that, for $t > \tau$, either

$$|(D_h^- u^h(t))_l - \mathrm{T}\left((D_h^- u^h(t))_k\right)| \leq C_1(\tau)(h|l-k|)^{1/4}, \tag{4.52}$$

or

$$|(D_h^- u^h(t))_l - (D_h^- u^h(t))_k| \leq C_1(\tau)(h|l-k|)^{1/4}, \tag{4.53}$$

where the exponent $1/4$ is due to the fact that the local inverses of W' are $1/2$-Hölder continuous, since

$$W'''(p^+) = -W'''(p^-) > 0.$$

Let $t > \tau$; observe that if $(D_h^- u^h(t))_k < 0$ then

$$\begin{aligned} &(4.53) \ \text{ holds when } \ (D_h^- u^h(t))_l < 0 \\ \text{and} \quad &(4.52) \ \text{ holds when } \ (D_h^- u^h(t))_l > 0. \end{aligned} \tag{4.54}$$

Similarly, if $(D_h^- u^h(t))_k > 0$ then

$$\begin{aligned} &(4.53) \ \text{ holds when } \ (D_h^- u^h(t))_l > 0 \\ \text{and} \quad &(4.52) \ \text{ holds when } \ (D_h^- u^h(t))_l < 0. \end{aligned} \tag{4.55}$$

Let now K be a compact subset of (ξ_i, ξ_{i+1}). Denote, as usual, by $\sigma_i = \frac{n_i}{m_i} \in \mathbb{Q}$ the value of $\overline{\varrho}$ on (ξ_i, ξ_{i+1}). Possibly reducing $h > 0$, we may assume that $\mathrm{dist}(K, \{\xi_i, \xi_{i+1}\}) > n_i h$. Recall also that by Definition 4.4.6

$$\left(D_{n_i h}^- u^h(t)\right)_k = \frac{1}{n_i} \sum_{l=k-n_i+1}^{k} \left(D_h^- u^h(t)\right)_l. \tag{4.56}$$

[8]For any $y \in (W'(p^+), W'(p^-))$, ${W'}^{-1}(y)$ consists of three elements, two of them in $[M^-, p^-] \cup [p^+, M^+]$ and the other one in (p^-, p^+).

By our choice of the admissible sequence $(\overline{u}^h)$ in Section 4.4.2 and remembering (4.20), it follows that among the terms $(D_h^- u^h(t))_l$ on the right-hand side of (4.56), m_i are negative, while $(n_i - m_i)$ are positive. Using (4.53), (4.54) and (4.55) we can then replace in (4.56) the terms $(D_h^- u^h(t))_l$ with m_i times the term $(D_h^- u^h(t))_k$ and with $(n_i - m_i)$ times the term $\mathrm{T}((D_h^- u^h(t))_k)$ at the expenses of an error of order $h^{1/4}$, and we obtain[9] for $t > \tau$,

$$\begin{aligned}
&\left|(D_{n_i h}^- u^h(t))_k - \frac{m_i (D_h^- u^h(t))_k + (n_i - m_i)\mathrm{T}\left((D_h^- u^h(t))_k\right)}{n_i}\right| \leq C_2(\tau) h^{1/4} \\
&\text{if } (D_h^- u^h(t))_k < 0 \\
&\left|(D_{n_i h}^- u^h(t))_k - \frac{(n_i - m_i)(D_h^- u^h(t))_k + m_i \mathrm{T}\left((D_h^- u^h(t))_k\right)}{n_i}\right| \leq C_2(\tau) h^{1/4} \\
&\text{if } (D_h^- u^h(t))_k > 0,
\end{aligned} \tag{4.57}$$

for a suitable constant $C_2(\tau) > 0$, for all $k \in \mathbb{N}$ such that $kh \in K$. From (4.52) and (4.53) we finally get our desired estimate on the averaged gradients:

$$|(D_{n_i h}^- u^h(t))_l - (D_{n_i h}^- u^h(t))_k| \leq C_3(\tau)(h|l-k|)^{1/4}, \qquad t > \tau. \tag{4.58}$$

for all $l, k \in \mathbb{N}$ such that $lh, kh \in K$, for a suitable constant $C_3(\tau)$.

We now want to apply the Ascoli-Arzelà Theorem; notice however that $D_{n_i h}^- u^h(t) \in PC_h((\xi_i, \xi_{i+1}))$, so that $D_{n_i h}^- u^h(t)$ is not continuous in general. Define the sequence

$$(p_h) \subset PL_h((\xi_i, \xi_{i+1}) \times (0, +\infty))$$

so that $p_h(t)$ is the linear interpolation of the values of $D_{n_i h}^- u^h(t)$ on the grid of size h, for any $t \in (0, +\infty)$. From (4.58), Lemma 4.6.6 and from the Ascoli-Arzelà Theorem (invading (ξ_i, ξ_{i+1}) with a sequence of compact subsets) it follows that (p_h) has a (not relabelled) subsequence converging to some

$$p \in \mathcal{C}((\xi_i, \xi_{i+1}) \times (0, +\infty))$$

uniformly on compact subsets of $(\xi_i, \xi_{i+1}) \times (0, +\infty)$ as $h \to 0^+$. Hence $(D_{n_i h}^- u^h)$ also converges to p uniformly on compact subsets

[9] To understand the meaning of inequalities (4.57), assume that the right hand sides vanish. Then, remembering (4.24), it follows that the first inequality of (4.57) says that $D_h u^h(t) = q(D_{n_i h}^- u^h(t), \sigma_i)$, while the second inequality says that $\mathrm{T}(D_h u^h(t)) = q(D_{n_i h}^- u^h(t), \sigma_i)$.

of $(\xi_i, \xi_{i+1}) \times (0, +\infty)$ as $h \to 0^+$, and therefore

$$p = u_x.$$

□

The origin of the term $q(u_x, \overline{\varrho})$ in equation (4.9) is then contained in the following result.

Corollary 4.6.8. *Under the assumptions of Theorem 4.6.7 we have that (u^h) has a (not relabelled) subsequence such that*

$$\lim_{h \to 0^+} W'\left(D_h^- u^h\right) = W'\left(q(u_x, \overline{\varrho})\right)$$

uniformly on compact subsets of $(\mathbb{T} \setminus \{\xi_1, \dots, \xi_M\}) \times (0, +\infty)$.

Proof. Fix $i \in \{1, \dots, M\}$ and a compact set $K \subset (\xi_i, \xi_{i+1})$. Recalling the notation in the proof of Theorem 4.6.7, and also (4.57), we have

$$\begin{cases} \left|q\left(D_{n_i h}^- u^h(t), \sigma_i\right) - D_h^- u^h(t)\right| \leq C(\tau) h^{1/4} & \text{if } D_h^- u^h(t) < 0, \\ \\ \left|q\left(D_{n_i h}^- u^h(t), \sigma_i\right) - \mathrm{T}(D_h^- u^h(t), \sigma_i)\right| \leq C(\tau) h^{1/4} & \text{if } D_h^- u^h(t) > 0. \end{cases}$$

for all $t > \tau > 0$ and a constant $C(\tau) > 0$. The assertion then follows from Theorem 4.6.7 and the continuity of W'. □

4.6.2 Limit evolution law

Theorem 4.6.7 is the crucial result which allows to pass to the limit as $h \to 0^+$ in the semidiscrete scheme (4.6).

Theorem 4.6.9. *Let $\overline{u}$, $\overline{\varrho}$, $(\overline{u}^h)$, (u^h) and $u \in \mathcal{X}$ be as in Theorem 4.6.7. Then u is a distributional solution of the following problem:*

$$u_t = \Big(W'\left(q\left(u_x, \overline{\varrho}\right)\right)\Big)_x \qquad \text{in } \mathbb{T} \times (0, +\infty). \tag{4.59}$$

Proof. Let $\varphi \in \mathcal{C}_0^1(\mathbb{T} \times (0, +\infty))$. Define $\varphi^h \in \mathcal{C}^1((0, +\infty); PL_h(\mathbb{T}))$ as the function having the same values as φ on the nodes of the grid. We multiply the first equation in (4.6) by φ^h. After an integration by parts we get

$$\int_0^\infty \sum_{i=1}^{N_h} u_i^h \, {\varphi_i^h}_t \, dt = \int_0^\infty \int_{\mathbb{T}} W'(D_h^- u^h) \, D_h^- \varphi^h \, dx \, dt.$$

Since (u^h) converges to u uniformly (see (4.47)) and φ_t^h converges uniformly to φ_t as $h \to 0^+$, we have

$$\lim_{h\to 0^+} \int_0^\infty \sum_{i=1}^{N_h} u_i^h \, \varphi_{i\,t}^h \; dt = \int_0^\infty \int_{\mathbb{T}} u\varphi_t \; dxdt. \tag{4.60}$$

Using Corollary 4.6.8 we get

$$\lim_{h\to 0^+} \int_0^\infty \int_{\mathbb{T}} W'(D_h^- u^h)\, D_h^- \varphi^h \; dx\, dt = \int_0^\infty \int_{\mathbb{T}} W'(q(u_x, \overline{\varrho}))\, \varphi_x \; dx\, dt. \tag{4.61}$$

From (4.60) and (4.61) we get

$$\int_0^\infty \int_{\mathbb{T}} u\, \varphi_t \; dxdt = \int_0^\infty \int_{\mathbb{T}} W'(q(u_x, \overline{\varrho}))\, \varphi_x \; dx\, dt,$$

which is the distributional formulation of (4.59). □

Concerning Theorem 4.6.9, some observations are in order.

Remark 4.6.10. Since $q(\cdot, \sigma)$ is increasing, equation (4.59) is well-posed and corresponds to the gradient flow in $L^2(\mathbb{T})$ of the convex functional

$$v \to F_{\overline{\varrho}}(v) := \int_{\mathbb{T}} f(v_x, x) \; dx, \tag{4.62}$$

where the Carathéodory integrand f is given by

$$f(p, x) := \int_0^p W'(q(s, \overline{\varrho}(x))) \; ds. \tag{4.63}$$

Notice that we can assume that f is defined on $\mathbb{R} \times \mathbb{T}$, by extending q to a continuous function on the whole of $\mathbb{R} \times [0, 1]$, increasing in the first variable. For instance, we may set

$$\begin{aligned} q(p, \sigma) &:= p^- \qquad && \text{if } p > \sigma p^- + (1-\sigma)M^+ \\ q(p, \sigma) &:= M^- \qquad && \text{if } p < \sigma M^- + (1-\sigma)p^+. \end{aligned}$$

By the results in [29], it then follows that (4.59) admits a unique solution for any initial datum $\overline{u} \in \mathcal{X} \subset L^2(\mathbb{T})$. Therefore, we can remove the extraction of a subsequence in Theorems 4.6.7 and 4.6.9. Let us also observe that $F_{\overline{\varrho}}$ has a one-parameter family of minimizers given by

$$x \in \mathbb{T} \to \int_0^x \Big(\sigma(s)p + (1-\sigma(s))\mathrm{T}(p)\Big) \; ds + \lambda,$$

where $p \in [M^-, p^-]$ is uniquely determined by the condition

$$\int_{\mathbb{T}} \Big(\sigma(s) p + (1 - \sigma(s)) \mathrm{T}(p) \Big) \, ds = 0$$

and $\lambda \in \mathbb{R}$.

Remark 4.6.11. Recalling (4.41), so that $u_t(\cdot, t) \in L^2(\mathbb{T})$ for almost every $t \in (0, +\infty)$, we have

$$\Big(W'\big(q\big(u_x(\cdot, t)\big), \overline{\varrho}(\cdot)\big) \Big)_x \in L^2(\mathbb{T}) \qquad \text{for a.e. } t \in (0, +\infty).$$

This implies in particular that $W'\big(q(u_x(\cdot, t)), \overline{\varrho}(\cdot)\big)$ is continuous on $\mathbb{T}$ for almost every $t \in (0, +\infty)$, and the following interior conditions hold:

$$W'(q(u_x(\xi_i^+, t), \sigma_i)) = W'(q(u_x(\xi_i^-, t), \sigma_{i-1})), \, t \in (0, +\infty), \\ i \in \{1, \ldots, M\}, \tag{4.64}$$

where $u_x(\xi_i^\pm, t)$ are the left and right limits of $u_x(\cdot, t)$ at the point ξ_i. From Remark 4.3.7 it also follows that if $\overline{u}_x(x) \in \mathrm{int}(G(\sigma_i))$ for any $x \in (\xi_i, \xi_{i+1})$, then

$$u \in \mathcal{C}^\infty((\xi_i, \xi_{i+1}) \times (0, +\infty)).$$

Remark 4.6.12. If $\Sigma_L(\overline{u}) = \emptyset$ it is natural to choose (among the infinitely many possible choices of $\overline{\varrho}$) either $\overline{\varrho} \equiv 0$ or $\overline{\varrho} \equiv 1$ in $\mathbb{T}$. Then equation (4.59) reduces to equation (4.2) which, in this case, is well-posed. If $\overline{u}$ and $\overline{\varrho}$ are as in Example 4.4.3, the regions of $\Sigma_L(u(t))$ evolve under (4.59) with $\overline{\varrho} = 1/2$. In particular in those regions we have $u_t \neq 0$ (unless u is linear). This is a different evolution with respect to the one numerically observed in [14] obtained as a limit of (4.5) as $\epsilon \to 0$.

Remark 4.6.13. A solution to (4.59) typically decreases the functional (4.62) (see Remark 4.6.10) but does not necessarily decreases the functional (4.1).

4.6.3 The case of a general $\overline{\varrho}$

In this last section we generalize Theorem 4.6.9, removing the assumption of $\overline{\varrho}$ being piecewise constant with rational values.

Theorem 4.6.14. *Let $\overline{u} \in \mathrm{Lip}(\mathbb{T})$ and assume that there exists $\overline{\varrho} \in L^\infty(\mathbb{T}; [0, 1])$ such that (4.27) holds. Then there exists an admissible sequence $(\overline{u}^h)$ for $\overline{u}$ such that the corresponding sequence (u^h) of solutions to (4.6) converges uniformly on compact subsets of $\mathbb{T} \times [0, +\infty)$ to a function $u \in \mathcal{X}$ which is a distributional solution of (4.59).*

Proof. Let $(\overline{\varrho}_n) \subset L^\infty(\mathbb{T}; \mathbb{Q} \cap [0,1])$ be a sequence of prc-percentages for $\overline{u}$ pointwise converging to $\overline{\varrho}$ as $n \to +\infty$, and let $\overline{u}_n^h$ be the admissible sequence for $\overline{u}$ constructed in Section 4.4.2 with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$. Let also u_n^h be the solutions to (4.6) with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$ and initial condition $\overline{u}_n^h$. From Theorem 4.6.9 it follows that (u_n^h) converges to a function $u_n \in \mathcal{X}$ solving distributionally (4.59) with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$. We will show that (u_n) converges to a distributional solution u of (4.59) as $n \to +\infty$. Indeed, let f_n be the function defined in (4.63), with $\overline{\varrho}$ replaced by $\overline{\varrho}_n$. By the pointwise convergence of $\overline{\varrho}_n$ to $\overline{\varrho}$, for almost every $x \in \mathbb{T}$ we have that $f_n(p,x) \to f(p,x)$ as $n \to +\infty$, locally uniformly in $p \in \mathbb{R}$. Therefore, the convex functional $F_{\overline{\varrho}_n}$ defined in (4.62) Γ-converges to $F_{\overline{\varrho}}$ in $L^2(\mathbb{T})$. Since these functionals are convex, by the results in [43] (see also [39, Th. 2.17]) it follows that the gradient flow u_n of $F_{\overline{\varrho}_n}$ converges in $\mathcal{C}^0([0,+\infty); L^2(\mathbb{T}))$ to the gradient flow u of $F_{\overline{\varrho}}$, which is the unique solution to (4.59) with initial datum $\overline{u}$. As the functions u_n are uniformly bounded in $\mathcal{X}$, it follows that $u \in \mathcal{X}$ and (u_n) converges to u uniformly on compact subsets of $\mathbb{T} \times [0,+\infty)$.
By a diagonal argument, we can eventually find a sequence (h_n) converging to 0^+ such that (u_n^h) converges to u uniformly on compact subsets of $\mathbb{T} \times [0,+\infty)$, thus concluding the proof. $\square$

Remark 4.6.15 (**Long time behavior**). Since (4.59) is the gradient flow of the convex functional $F_{\overline{\varrho}}$ in $L^2(\mathbb{T})$, by [29] (see also [5]) it follows that there exists

$$\lim_{t \to +\infty} u(t),$$

where the limit is taken in $L^2(\mathbb{T})$, and hence in $L^\infty(\mathbb{T})$. Moreover, recalling (4.17), such a limit is the global minimizer of $F_{\overline{\varrho}}$ in $L^2(\mathbb{T})$ with the choice

$$\lambda = \int_{\mathbb{T}} \overline{u}\, dx.$$

Chapter 5

Many-particle interactions

In this chapter, we extend the nearest-neighbour interaction model to an interaction with finitely many particles. We find out that the continuum limit is still the stochastic Allen-Cahn equation.

The proof is a straight-forward extension of the result obtained by Bensoussan and Temam [18] and Pardoux [101] to approximating schemes modeling the interaction between a large, but finite number of particles.

In order to reduce confusion and to provide a self-contained description within the chapter, we will repeat some constructions done in this chapter also in the next chapter, where we study long-range interactions. We will also point out differences in the objects (in particular the choice of the interaction weights) between this chapter and the next chapter.

5.1 Introduction and notation

We consider a system of N coupled particles on a lattice $\Lambda = \mathbb{Z}/N\mathbb{Z}$. Each particle is subject to a force derived from a bistable potential V and perturbed by Brownian noise. The particle system can be described as a vector-valued function $u^N = (u_1^N, u_2^N, \dots, u_N^N)$ which is a solution of the system of N coupled stochastic differential equations

$$\begin{aligned} du_i^N(t) = \tilde{\gamma} \sum_{j=-R}^{R} J(j) \left(u_{i+j}^N(t) - u_i^N(t)\right) dt \\ - V'(u_i^N(t))dt + \sqrt{2\sigma} d\widetilde{B}_i(t),\ i \in \Lambda \end{aligned} \tag{5.1}$$

with initial condition $u^N(0) \in \mathbb{R}^N$. Here, $u_i^N(t)$ are the components of the vector $u^N(t) \in \mathbb{R}^N$, $J(j) \in \mathbb{R}_+$ are weights, $V(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2$ and $\widetilde{B}_i$ are independent Brownian motions. $\tilde{\gamma}$ is a constant and $\sqrt{2\sigma}$ the intensity of the noise.
In this model, each particle interacts with all of its neighbours up to distance R and R remains finite (though probably very large). The weights $J(j)$ describes the strength of the interaction between two particles at site i and $i+j$.
We want to prove that under certain conditions on the weights $J(j)$ the limit of solutions u^N to (5.1) converges as $N \to \infty$, to a well-posed stochastic PDE,

$$\begin{aligned}
\partial_t u(x,t) &= \gamma A u(x,t) - V'(u(x,t)) \; + \frac{\partial^2}{\partial_x \partial_t} W(x,t), \; (x,t) \in [0,1] \times \mathbb{R}^+ \\
u(0,t) &= u(1,t) \\
u(0,.) &= u_0,
\end{aligned} \tag{5.2}$$

where A is a second-order differential operator, $\gamma > 0$ is the diffusion constant, V a double well potential, and $\frac{\partial^2}{\partial_x \partial_t} W(x,t)$ denotes space-time white noise.
In Section 5.2 we propose a condition on the weights such that the continuous operator A in (5.2) is the Laplacian, which means that (5.2) is the stochastic Allen-Cahn equation.
Our proof of convergence of a suitably rescaled version of (5.1) to (5.2) is done via Galerkin approximation. A priori bounds on the approximating solutions in a suitable Banach space are established. When passing to the limit, we can make use of the theory of nonlinear monotone operators in order to identify the limit operators and thus the limit equation.
Monotone operators are a classical tool in nonlinear functional analysis, which have been introduced by Browder and Minty in the 1960s. Following first results by Bensoussan and Temam [18], Pardoux systematically showed applications of the theory of monotone operators to SPDEs in his Ph.D. thesis [101]. Due to the poor regularity of space-time white noise, most cases of monotone operators in SPDEs studied in the literature deal with equations is driven by multiplicative noise.

Notation We denote with $[0,T] \subset \mathbb{R}$ the time interval and with $D \subset \mathbb{R}$ the space interval. We set $D = [0,1]$ throughout this chapter, but we often use D to prevent any confusion.

Probability space The probability space is denoted by $(\Omega, \mathcal{F}, \mathbb{P})$ and B_t denotes a one-dimensional Brownian Motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Banach Spaces Denote by $\|.\|$ the $L^2(D)$-norm and by $(.,.)$ the corresponding inner product. We define the Sobolev norm

$$\|u(x)\|_{W^{1,p}(D)} := \left(\int_D \left(|u(x)|^p + |\nabla u(x)|^p \right) dx \right)^{1/p}$$

and the Sobolev spaces $W^{1,p}(D)$ as the completion of $C^\infty(D)$ with respect to the $W^{1,p}(D)$-norm. For $p = 2$ we get the Hilbert Space $W^{1,2}(D) \equiv H^1(D)$. By $H^{-1}(D)$ we indicate the dual of the space $H_0^1(D)$. Denote by $\langle .,. \rangle$ the duality pairing between H^1 and H^{-1}. Note that the Banach spaces $H_0^1 \subset L^2 \subset H^{-1}$ form a Gelfand triple, meaning that the embeddings are continuous and dense.
We denote with $Y_T := L^2(\Omega, C([0,T], L^2(D))) \cap L^2(\Omega \times [0,T], H_0^1(D))$ the reflexive Banach space with norm

$$\|u(t,x,\omega)\|_{Y_T} := \left(\mathbb{E} \left[\sup_{t \in [0,T]} \|u(t,x,\omega)\|^2_{L^2(D)} + \int_0^T \|u(t,x,\omega)\|^2_{H_0^1(D)} dt \right] \right)^{1/2}. \tag{5.3}$$

We are looking for solutions to (5.2) in the space Y_T, which are generally called weak (in the PDE sense) solutions or variational solutions. We end this section with the definition of a weak (in the PDE sense) solution to an SPDE.

Definition 5.1.1 ([37], page 168)**.** An adapted V-valued stochastic process $\{u(.,t) : t \in [0,T]\}$, $u(.,t) : \Omega \to L^2(D)$ is called a **weak solution** of $du = (Au + f)dt + dW$ if
(a) $u \in L^2(\Omega \times [0,T]; V)$ (b) for all test functions $\phi(x) \in dom(A)$ the following equation

$$\begin{aligned}(u(T), \phi)_{L^2} = (u(0), \phi)_{L^2} &+ \int_0^T \langle Au(s), \phi \rangle ds \\ &+ \int_0^T (f(u), \phi)_{L^2} \, ds + \int_0^T (\phi, dW(s))_{L^2}\end{aligned} \tag{5.4}$$

holds for each $t \in [0,T]$ $\mathbb{P}$- almost surely.

We refer to Section 2.1.2 of Chapter 2 for some comments on this notion of solution and to the book [37] for a concise treaty.

5.2 The discrete-in-space equation

The interaction between the particles can be interpreted mathematically as a *finite difference operator*: Each particle $u_i^N(t)$ is interpreted as a nodal function, which describes the value $u_i(t)$ at the node i at time t of the piecewise linear function $u^h(t)$. Central difference operator use as information the nodal function (in time) $u_i(t)$ at node i as the center and the $2R$ neighbouring nodes u_j with $j \in \{-R, \ldots -1, 1, \ldots, R\}$ around this center node.
The standard construction of a central difference operator which approximates a second order differential operator (in our case the operator A from (5.2)) goes as follows:
Look at Taylor's approximation of a sufficiently regular function u around the point x_0, once with the right neighbour in a grid with grid size h

$$\begin{aligned} u(x_0+h) = u(x_0) + hu_x(x_0) + h^2\frac{u_{xx}(x_0)}{2!} + h^3\frac{u_{xxx}(x_0)}{3!} \\ + h^4\frac{u_{xxxx}(x_0)}{4!} + O(h^5) \end{aligned} \tag{5.5}$$

and once with the left neighbour

$$\begin{aligned} u(x_0-h) = u(x_0) - hu_x(x_0) + h^2\frac{u_{xx}(x_0)}{2!} - h^3\frac{u_{xxx}(x_0)}{3!} \\ + h^4\frac{u_{xxxx}(x_0)}{4!} - O(h^5). \end{aligned} \tag{5.6}$$

Now add (5.5) and (5.6) and choose $x_0 = ih$ to get

$$u_{i+1} + u_{i-1} = 2u_i + h^2 u_{xx} + 2h^4\frac{u_{xxxx}(x_0)}{4!} + O(h^6). \tag{5.7}$$

We obtain the simplest second order symmetric finite difference approximation to $u_{xx}(x_i)$

$$u_{xx}(x_i) \approx \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} \tag{5.8}$$

with error of order $O(h^2)$. Note that we exploited here the equidistant grid and the symmetry of the stencil to cancel the odd derivatives and raise the order of convergence.
The stencil (5.8) takes into consideration the two neighbouring nodes $i-1$ and $i+1$ of i and therefore models nearest neighbour interaction. By adding more neighbours in the same way, we can construct the central difference operator corresponding to the interaction of the

particle u_i with its $2R$ nearest neighbours. Each nodal function $u_j(t)$ is weighted with some weight $J(j)$, which determines the interaction strength on the particle at site j when the reference site is at the location of the particle i. A common choice for the constant in the denominator is the width of the symmetric stencil, i.e. the number of neighbouring nodes considered to the left (or right).

$$A_R^h u_i = \frac{J(R)u_R + \ldots + J(1)u_{i+1} - J(0)u_i + J(-1)u_{i-1} + \ldots + J(-R)u_{-R}}{Rh^2} \tag{5.9}$$

From the above construction we can also infer information on the accuracy or rate of convergence of a difference stencil to a continuous operator, assuming sufficient regularity of the solution up to the boundary of the domain. (Indeed, it is this high regularity restrictions which make finite difference schemes less attractive in numerical analysis). This is done via the order of *consistency* of the stencil. Again, we restrict to symmetric stencils by setting $J(j) = J(-j)$ and use the cancellation effects given by the symmetry of the stencil and the equidistant grid to get the following error estimate for A_R^h as in (5.9) with initial data $u_0 \in C^4([0,1])$

$$\sup_{y\in[0,1]} \left| A_R^h u(y) - u_{xx}(y) \right| \le O(h^2). \tag{5.10}$$

This estimate can be found in [115]. The hightest accuracy is obtained by performing the construction via Taylor's approximation as sketched above and adjust the denominator in such a way that some higher order terms cancel. As this is irrelevant for our purpose, we omit this discussion and refer the interested reader to [115] for details.

Assumptions on the interaction strength We now list more assumptions we need to make on the weights $J(j)$. For this it is useful to interpret the collection of weights $J(j)$ as a weight function

$$\begin{aligned} J : \mathbb{Z} &\to \mathbb{R}_+ \\ j &\mapsto J(j). \end{aligned} \tag{5.11}$$

The weight function J has the value $J(0)$ at the reference site i and the value $J(j)$ at the particle j at distance j from the reference site. We set $J(j) = 0$ for all $j > R$.

Assumption 5.2.1 (Assumptions on the weights)**.** *We assume that the weight function $J : \mathbb{Z} \to \mathbb{R}_+$ satisfies the following properties:*

- *J is symmetric*

- *$J(j)$ is positive and finite*
- *J satisfies the diagonal dominance relation*

$$J(0) = 2\sum_{j=1}^{R} J(j) \tag{5.12}$$

The use of the symmetry assumption was explained in the above paragraph.
The positivity assumption is a matter of convenience, but is not strictly necessary as long as the diagonal dominance relation holds. The relevant assumption (and restriction) on our weight function is a diagonal dominance relation $J(0) \geq 2\sum_{j=1}^{R} J(j)$, which we reinforce to a diagonal sum property for convenience. Rewriting the central difference operator A_R^h from (5.9) as

$$-A_R^h u(ih) = \frac{1}{Rh^2}\left(J(0)u_i - \sum_{j=-R, j\neq 0}^{R} J(j)u_{i+j}\right) \tag{5.13}$$

we see that the exact relationship between $J(0)$ and $\sum_{j=1}^{R} J(j)$ decides about the definiteness of the matrix A_R^h: indeed, $A_R^h u(ih)$ is the i-th entry of the N-dimensional vector $A_R^h u^h$ and the difference operator A_R^h can be seen, for every fixed $N = \frac{1}{h}$, as the $N \times N$-dimensional band matrix with entries $J(0)$ on the diagonal and $J(j)$ on the $j-th$ subdiagonal. As J is symmetric, the entries on the superdiagonals are the same as on the subdiagonals, hence A_R^h is symmetric.
If $J(0) \geq \sum_{j=1}^{R} J(j)$ in each row, then $-A_R^h$ is called diagonally dominant, which implies that the matrix $-A_R^h$ is positive definite.
Allowing the exact equality (5.12) corresponds to positive semidefiniteness. The convenience of (5.12) is that it allows to write the difference operators A_R^h in the general R-nearest-neighbour form

$$A_R^h u(ih) \overset{(5.12)}{=} \frac{\gamma}{Rh^2}\sum_{j=-R}^{R} J(j)\left(u_{i+j}(t) - u_i(t)\right). \tag{5.14}$$

Note and comparison with Chapter 6 Note that we did not make any assumption on the integrability of J, as we don't pass to the limit as $R \to \infty$ in this chapter. The integrability assumption would be needed in that case to ensure that the stencil remains coercive when R is growing. Though it is not necessary from the mathematical point of view as long as (5.12) and the monotonicity assumption holds, it seems natural to choose a decaying and summable weight function. For the choices such as $J(j) = \frac{1}{j^2}$, we have

$$\frac{1}{R}\sum_{j=1}^{R} j^2 J(j) = 1 \tag{5.15}$$

which is a discrete version of the second moment condition (6.6) in Chapter 6.
Due to the usage of the rescaled weight function $J_R(j) = J\left(\frac{j}{R}\right)$ in Chapter 6, the discrete version of the second moment condition then reads $\frac{1}{R^3}\sum_{j=-R}^{R} J_R(j)j^2 = 1$, see (6.10) in the proof of Lemma 6.2.1.

Projection operators For the convenience of the reader, we state the projection operators used in the Galerkin approximation. This also helps to formalize our usage of the L^2-scalar product also in the discrete case, instead of switching to the summation notation.
Let $\{v_k\}$ be an orthonormal basis of L^2 with $\{v_k\} \subset H_0^1$, and span $\mathbb{R}^N$ with the first N basis vectors. Suppose that $P_N : L^2 \to \mathbb{R}^N$ is an orthogonal projection operator such that

$$P_N f = \sum_{k=1}^{N} (f, v_k)\, v_k \qquad \text{for all } f \in L^2, \tag{5.16}$$

Via this projection operator, we define $V'(u^h) := P_N V'(u)$ and $W^h(t) := P_N W(x,t)$. We extend P_N to a projection operator $P'_N : H^{-1} \to \mathbb{R}^N$ defined by

$$P'_N w = \sum_{k=1}^{N} \langle w, v_k\rangle_{H^{-1},H_0^1} v_k \qquad \text{for all elements } w \in H^{-1}, \tag{5.17}$$

Via $A^h = P'_N A$, we can consider the finite difference matrix A_R^h defined in (5.9) as the projection of a linear operator $A : H_0^1 \to H^{-1}$.

From particle systems to finite difference approximations The particle system (5.1) is giving us the “microscopic” evolution in terms of particles, while the SPDE (5.1) displays a “macroscopic” evolution of the system. Therefore, some rescaling procedures are necessary to transform (5.1) into a discrete-in-space approximation scheme: We rescale the unit lattice $\Lambda = \mathbb{Z}/N\mathbb{Z}$ by $h = \frac{1}{N}$ to arrive at the uniform grid $D_h = \{0, h, \ldots, Nh\}$ where we identify $0 = 1 = Nh$. D_h is then a discretization of the interval $[0,1]$ in equidistant nodes

and periodic boundary conditions. We call h the grid size and will sometimes refer to ih as the "node i".
Moreover, we rescale the coupling constant $\tilde{\gamma}$ by h^{-1} and the potential term by h. Then we accelerate time by a factor $\frac{1}{h}$, i.e. we set $\widetilde{X}(t) = X(t/h)$, which gives us another extra h^{-1} on the coupling constant and cancels out the previous changes in the scaling of the potential. Moreover, this acceleration of time gives us a different sequence of independent Brownian motions, which we call $B_i(t)$. The real-valued stochastic process $\widetilde{X}_i^h(t)$ can then be identified with the real-valued function $u_i(t,\omega)$ of nodal values at the node i.
The resulting rescaled system of SDEs reads for all $i = 1 \dots N$

$$du_i(t) = \left(\frac{\gamma}{Rh^2} \sum_{j=-R}^{R} J(j) \left(u_{i+j}(t) - u_i(t) \right) \right) dt - V'(u_i(t))dt + \sqrt{\frac{2\sigma}{h}} dB_i(t). \tag{5.18}$$

Note that $u_i(t)$ is defined only at one specific node i.
Via $u_i(t) := u^h(ih,t)$, the vector-valued function of nodal values $u^h(t) = (u_1(t), u_2(t), \dots, u_N(t))$ on the grid D_h can be identified as a continuous, piecewise linear function on $u^h(x,t) : D \times \mathbb{R}^+ \to \mathbb{R}$.

Comparison with Chapter 6 The discrete-in-space system (5.18) is the same as the discrete system (6.3) in Chapter 6. The different scaling factor $\frac{\gamma}{Rh^2}$ in R comes from the fact that we use a quadratically decaying weight function J here, while in Chapter 6 the weights don't have such a decay.
The system (5.18) reads in integral form:

$$\begin{aligned} u_i(t) \ = \ u_i(0) \ + \ \frac{\gamma}{Rh^2} \int_0^t \sum_{j=-R}^{R} J(j) \left(u_{i+j}(s) - u_i(s) \right) ds \\ - \int_0^t V'(u_i(s)) ds \ + \int_0^t dB_i(s). \end{aligned} \tag{5.19}$$

Recalling (5.14), we write the approximative equation as a discrete in space evolution equation with the difference operator A_R^h

$$\frac{d}{dt} u^h(t) \ = \gamma A_R^h u^h dt - V'(u^h)dt \ + dW^h(t) \tag{5.20}$$

where $W^h(t) = [W(t,x_1), \dots, W(t,x_{N-1})]^T$ with $x_i = ih$ is the vector-valued function of Brownian Motions B_i on the nodes. We refer the interested reader to [84] for details on the numerical approximation of the white noise.

Note that, for finite N, (5.20) is an Ito-equation in $\mathbb{R}^N$ and we see that the coefficients $P_N V'(u) = V'(u^h)$ are locally bounded, locally Lipschitz continuous and monotone. Therefore, a solution to (5.20) exists for some time interval $[0,T]$ and satisfies $u^h \in L^2(\Omega \times [0,T], H_0^1(D)) \cap L^2(\Omega; C([0,T], L^2(D)) \subset Y_T$, where Y_T is the Banach space defined in the introduction of this chapter, see (5.3).

5.3 A priori estimates

In this section, we calculate a priori estimates on the discrete-in-space evolution (5.20). As the weights $J(j)$ are fixed numbers and do not change with time, the central difference operator $-A_R^h$ is deterministic and time-independent.
As we are dealing with monotone operators (see also Lemma 5.3.1), we state first the relevant properties, which can be found, for example, in [119] or [102].

5.3.1 Properties of the deterministic operators

In the following we state some properties of the operators A and $-V'$ which we use in the proof. We use the notation $\Omega_T := \Omega \times [0,T]$.

(i) *coercivity*: There exists $\alpha \in (0,\infty)$ and $\beta \in \mathbb{R}$ such that for all $u \in H_0^1$ the following inequality holds

$$\begin{aligned} \gamma \langle Au, u\rangle_{H_0^1} \;-\; (V'(u),u)_{L^2} \;\le\; -\,\underline{\alpha}\|u\|^2_{H_0^1} + \beta\|u\|^2_{L^2} \\ \text{for a.e. } (t,\omega)\in\Omega_T \end{aligned} \tag{5.21}$$

(ii) *strong monotonicity*: There exists $\delta \in \mathbb{R}$ such that for all $u, v \in H_0^1$

$$\langle A(u-v), u-v\rangle_{H_0^1} \;-\; (V'(u)-V'(v), u-v)_{L^2} \;\le\; \delta\|u-v\|^2_{L^2} \tag{5.22}$$

(iii) *hemicontinuity* A is weakly continuous on the affine lines of H_0^1, in formulas:

$$\begin{aligned} \theta \to \langle B(u+\theta v), w\rangle \quad \text{is continuous from } \mathbb{R}\to\mathbb{R} \\ \text{for all } u,v,w \in H_0^1 \end{aligned} \tag{5.23}$$

Moreover, the following local growth conditions are satisfied for the sum of operators $A - V'$ and used in subsequent proofs:

- For some finite real number $m > 0$ exists C_m such that for any u with $\|u\|_{H_0^1} < m$ holds

$$|(V'(u), u)_{L^2}| \leq C_m(1+\|u\|_{H_0^1}^2) \qquad \text{for a.e. } (t,\omega) \in \Omega_T \quad (5.24)$$

  This condition ensures that the boundedness of u^h implies the weak convergence of a subsequence $(A^h - V')u^h$ as $h \to 0$.

- Local Lipschitz continuity in H_0^1: Can be calculate thanks to the Sobolev embedding theorem, which gives that the embedding $L^4 \hookrightarrow W^{1,2}$ is continuous (in dimension $d \leq 3$). Therefore we have the estimate $\|u\|_{L^4}^4 \leq \|u\|_{H_0^1}^4$ and $\|u^2 v\|^2 \leq c\|u\|_{H_0^1}^4 \|v\|_{H_0^1}^2$. Using those, we can calculate

$$\begin{aligned} \|V'(u) - V'(v)\|_{L^2} &= \left\|u^3 - v^3\right\|_{L^2}^2 \\ &\leq C(\|u\|_{H_0^1}^4 + \|v\|_{H_0^1}^2)\|u-v\|_{H_0^1}^2 \end{aligned} \tag{5.25}$$

  This is used to prove a (local-in-time) existence of a solution to the finite dimensional problem.

Note that by projection and the boundedness of the sum of weights $\sum_j J(j)$, all properties hold also for the discrete operators A_R^h and $V'(u^h)$.

Notes Note that strong monotonicity implies coercivity, as shown in [119], page 501. That makes the coercivity property superfluous in some sense, but we stated it nevertheless as we have employed in several estimates in the sequel.
To see that our sum of operators are indeed monotone, we state the following statement from Pardoux [102], page 37.

Lemma 5.3.1. *Operators of the form $Au(x) + f(u(x))$ where A is a monotone operator and $f : \mathbb{R} \to \mathbb{R}$ is the sum of a Lipschitz and a decreasing function, are monotone.*

Indeed, we see that the monotonicity is preserved if we add a Lipschitz function, as a Lipschitz continuous term is absorbed in the RHS of the defining inequality (5.22). Decreasing functions preserve the 'direction' of the inequality.
Again, we note that as V' is only locally Lipschitz continuous, monotonicity is ensured only up to a certain time T, and therefore existence of a solution to (5.2) is guaranteed a priori only up to time T. The extension is though straightforward thanks to the global existence result for the continuous limit equation.

5.3.2 Uniform estimates

In this section, we derive suitable compactness properties on the sequence of solutions u^h of (5.20), which are needed to pass to the limit.

Proposition 5.3.2. *Let the initial data $u_0(x,\omega)$ be deterministic. The discrete solution u^h of* (5.20) *is uniformly bounded in the Y_t norm,i.e.*

$$\sup_h \mathbb{E}\left[\sup_{t\in[0,T]} \|u^h(t)\|^2_{L^2(D)} + \int_0^T \|u^h(t)\|^2_{H^1_0(D)}dt\right] < \infty. \tag{5.26}$$

Proof of Proposition 5.3.2. It is convenient to consider $u^h(\cdot,t,\omega)$ in the basis v_k we considered in (5.17), i.e. an ONB of L^2 made of elements of H^1_0. We intend such an u^h whenever we refer to the approximated problem in integral form:

$$u^h(t) \;=\; u^h_0 \;+\; \gamma\int_0^t A^h_R u^h(s)\,ds \;-\; \int_0^t V'(u^h(s))\,ds \;+W(t). \tag{5.27}$$

By Ito's formula (see [101], Chapter 2, for a suitable version),

$$\begin{aligned}\|u^h(t)\|^2_{L^2} \;=\; \|u^h(0)\|^2_{L^2} \;+\; 2\gamma\int_0^t \left\langle A^h_R u^h(s), u^h(s)\right\rangle_{H^1_0} ds \\ -\;2\int_0^t \left(V'(u^h(s)), u^h(s)\right)_{L^2} ds \;+\; 2\int_0^t \left(u^h(s), dW^h(s)\right)_{L^2}.\end{aligned} \tag{5.28}$$

From this above equation we derive the following bounds:

Boundedness in $L^2(\Omega\times[0,T], H^1_0(D))$ Using the coercivity of the bilinear form (5.21), we get from (5.28)

$$\begin{aligned}\|u^h(t)\|^2_{L^2} \;\le\; \|u^h(0)\|^2_{L^2} \;-\; 2\gamma\underline{\alpha}\int_0^t \|u^h(s)\|^2_{H^1_0}\,ds \\ +\,2\beta\int_0^t \|u^h(s)\|^2_{L^2}ds \;+\; 2\int_0^t \left(u^h(s), dW^h(s)\right)_{L^2}\end{aligned} \tag{5.29}$$

which is

$$\begin{aligned}\|u^h(t)\|^2_{L^2} \;+\; 2\gamma\underline{\alpha}\int_0^t \|u^h(s)\|^2_{H^1_0}\,ds \;\le\; \|u^h(0)\|^2_{L^2} \\ +\,2\beta\int_0^t \|u^h(s)\|^2_{L^2}ds \\ +\;2\int_0^t \left(u^h(s), dW^h(s)\right)_{L^2}.\end{aligned} \tag{5.30}$$

Taking the expectation, the last term $\int_0^t \left(u^h(s), dW^h(s)\right)_{L^2} = 0$ and we arrive at

$$\begin{aligned}\mathbb{E}\left[\|u^h(t)\|_{L^2}^2\right] \;+\; 2\gamma\underline{\alpha}\mathbb{E}\left[\int_0^t \|u^h(s)\|_{H_0^1}^2\, ds\right]& \\ \le\; \mathbb{E}\left[\|u^h(0)\|_{L^2}^2\right]& \\ +\; 2\beta\mathbb{E}\left[\int_0^t \|u^h(s)\|_{L^2}^2 ds\right]&.\end{aligned} \tag{5.31}$$

As we assumed that the initial data is deterministic, we can apply Gronwall's Lemma and get

$$\sup_h \sup_{0\le t\le T} \mathbb{E}\left[\|u^h(t)\|_{L^2}^2\right] \le c\|u^h(0)\|_{L^2}^2 \tag{5.32}$$

which is, noting that for piecewise linear functions u^h holds $\|u^h(0)\|_{L^2}^2 \le \|u_0\|_{L^2}^2$,

$$\sup_h \sup_{0\le t\le T} \mathbb{E}\left[\|u^h(t)\|_{L^2}^2\right] \le c\|u_0\|_{L^2}^2. \tag{5.33}$$

As $\|u_0\|_{L^2}^2 \le c$, (5.31) reduces to

$$\sup_h \mathbb{E}\left[\int_0^T \|u^h(t)\|_{H_0^1}^2\, ds\right] < \infty \tag{5.34}$$

which is the desired estimate.

Boundedness in $L^2(\Omega, C([0,T], L^2(D)))$ Starting again with the tested approximative problem (5.28), we now take first the supremum over t to get

$$\begin{aligned}\sup_{t\le T}\|u^h(t)\|_{L^2}^2 \;\le\; \|u^h(0)\|_{L^2}^2 \;+\; 2\gamma\int_0^t \left(A_R^h u^h(s), u^h(s)\right)_{L^2} ds& \\ -\; 2\int_0^t \left(V'(u^h(s)), u^h(s)\right)_{L^2} ds& \\ +\; 2\sup_{t\le T}\left|\int_0^t \left(u^h(s), dW^h(s)\right)_{L^2}\right|&.\end{aligned} \tag{5.35}$$

Applying coercivity (5.21), we arrive at

$$\begin{aligned}\sup_{t\le T}\|u^h(t)\|^2_{L^2} \;+\; 2\gamma\underline{\alpha}\int_0^T \|u^h(t)\|^2_{H^1_0}\,dt \\ \le\; \|u^h(0)\|^2_{L^2} + 2\beta\int_0^T \|u^h(t)\|^2 dt \\ +\; 2\sup_{t\le T}\left|\int_0^T \left(u^h(t), dW^h(t)\right)_{L^2}\right|.\end{aligned} \tag{5.36}$$

Now we use the BDG inequality to estimate the noise term:

$$\begin{aligned}\mathbb{E}\left[\sup_{t\le T}\left|\int_0^T \left(u^h(t), dW^h(t)\right)_{L^2}\right|\right] \;&\le\; c\,\mathbb{E}\left[\left(\int_0^T \left(u^h(t), u^h(t)\right)^2_{L^2} dt\right)^{1/2}\right] \\ &\le\; c\,\mathbb{E}\left[\|u^h(t)\|_{L^2(0,T;L^2(D))}\right].\end{aligned} \tag{5.37}$$

Note that $\|u^h(t)\|_{L^2(0,T;L^2(D))} \le \|u^h(t)\|_{L^2(0,T;H^1_0(D))} \le c$ from the previous step. Therefore, we finally arrive at

$$\mathbb{E}\left[\sup_{t\le T}\|u^h(t)\|^2_{L^2}\right] \;+\; 2\gamma\underline{\alpha}\mathbb{E}\left[\int_0^T \|u^h(t)\|^2_{H^1_0}\,dt\right] \le c \tag{5.38}$$

so in particular

$$\mathbb{E}\left[\sup_{t\le T}\|u^h(t)\|^2_{L^2}\right] \le c \tag{5.39}$$

with a constant c which is independent of h, which means that u^h is bounded in $L^2(\Omega, C([0,T], L^2(D)))$ independently of h. □

5.4 Convergence and identification of the limit

Proposition 5.4.1. *Let the initial data $u_0(x,\omega)$ be deterministic. Then, as $h \to 0$, solutions u^h of* (5.20) *converge in the following sense:*

$$\begin{aligned} u^h &\rightharpoonup u && \text{in } L^2(\Omega_T, H^1_0(D)) \\ u^h(T,.) &\rightharpoonup U && \text{in } L^2(\Omega, L^2(D)) \\ A^h_R u^h &\rightharpoonup \xi && \text{in } L^2(\Omega_T, H^{-1}(D)) \\ V'(u^h) &\rightharpoonup \zeta && \text{in } L^2(\Omega_T, L^2(D)) \end{aligned} \tag{5.40}$$

Proof of Proposition 5.4.1. We have proved in 5.3.2 that $u^h(t)$ is bounded uniformly in h in the reflexive Banach space Y_T. Therefore, we extract a subsequence, which we still denote by $u^h(t)$, which converges weakly in $L^2(\Omega \times [0,T], H_0^1(D))$ to some limit function $u(x,t)$. $\square$

Lemma 5.4.2. *Under the same assumptions as in 5.4.1, we have* $U(T) = u(T)$.

Proof of Lemma 5.4.2. As we know (5.40), it is left to identify the endpoint. To this aim, we extend the time interval to $]-\epsilon, T+\epsilon[$, and write down the discrete equation in integral form (5.19) for this extended open interval, and replace

$$u_i(T) = \big(u^h(T), e_i\big) = \big(u^h(T), e_i\big)\,\theta(t-T) \qquad \text{for } t \in]-\epsilon, T+\epsilon[, \; i = 1 \dots N \tag{5.41}$$

Thanks to the boundedness of all terms (5.40) we can pass to the limit as $h \to 0$ in equation (5.19) (where we set all terms to zero if $t \notin [0,T]$) and remain with

$$\big(u^h(T), e_i\big)\,\theta(t) - (U(T), e_i)\,\theta(t-T) \qquad \text{for } t \in]-\epsilon, T+\epsilon[, i = 1 \dots N, \tag{5.42}$$

which suffices to identify $U(T) = u(T)$. $\square$

Lemma 5.4.3. *For* $u \in L^2\big(\Omega \times]0,T[; H_0^1(D)\big) \cap L^2\big(\Omega; L^\infty(0,T,L^2(D)\big)$ *holds*

$$\mathbb{E}\left[\|u(T)\|_{L^2}^2\right] \le \liminf_{h\to 0} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2\right] \tag{5.43}$$

Proof of Lemma 5.4.3. As $u^h \rightharpoonup u$ in $L^2(D)$, by lower semicontinuity of the L^2-norm, $\|u(T)\|_{L^2} \le \liminf_{h\to 0} \|u^h(T)\|_{L^2}$. As the mapping $u \to \mathbb{E}\left[\|u\|_{L^2}^2\right]$ is convex as a map from $L^2(\Omega_T; L^2(D))$ to $\mathbb{R}$, we get furthermore for any $t \in [0,T]$, so in particular for the end point T,

$$\mathbb{E}\left[\|u(T)\|_{L^2}^2\right] \le \liminf_{h\to 0} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2\right] \tag{5.44}$$

by strong convergence of the initial condition,

$$\begin{aligned} \mathbb{E}\left[\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2\right] &= \mathbb{E}\left[\|u(T)\|_{L^2}^2\right] - \mathbb{E}\left[\|u_0\|_{L^2}^2\right] \\ &\le \liminf_{h\to 0} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2\right] - \liminf_{h\to 0} \mathbb{E}\left[\|u_0^h\|_{L^2}^2\right] \end{aligned}$$

which finally leads to

$$\mathbb{E}\left[\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2\right] \le \liminf_{h\to 0} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2 - \|u_0^h\|_{L^2}^2\right] \tag{5.45}$$

and the proof is concluded. $\square$

Theorem 5.4.4. *For any fixed R, as $h \to 0$, the solution u^h of the approximated problem* (5.20) *converges strongly to u in*

$L^2(\Omega\times]0,T[;H_0^1(D))$ *and in* $L^2(\Omega, C([0,T], L^2(D)))$. *Moreover, u solves* (5.2).

Proof of Theorem 5.4.4. First, we test (5.27) with functions $\phi(x) \in H_0^1(D)$:

$$\begin{aligned}\left(u^h(t), \phi(x)\right)_{L^2(D)} &= (u_0^h, \phi(x))_{L^2(D)} \\ &+ 2\gamma \int_0^t \left\langle A^h u^h(s,x), \phi(x)\right\rangle_{H_0^1(D)} ds \\ &- 2\int_0^t \left(V'(u^h(s,x)), \phi(x)\right)_{L^2(D)} ds \\ &+ 2\int_0^t \left(\phi(x), dW^h(s,x)\right)_{L^2(D)}.\end{aligned} \tag{5.46}$$

We pass to the limit in (5.46), use Lemma 5.4.2 and get that for all $\phi \in H_0^1(D)$

$$\begin{aligned}(u(T), \phi(x))_{L^2} &= (u_0, \phi(x))_{L^2} + 2\gamma \int_0^T \langle \xi, \phi(x)\rangle_{L^2} dt \\ &- 2\int_0^T (\zeta, \phi(x))_{L^2} dt + 2\int_0^T (\phi(x), dW(t,x))_{L^2}.\end{aligned} \tag{5.47}$$

It remains to identify the limit objects ξ and ζ and prove that u indeed solves (5.2).

Taking the expectation over (5.28), we get

$$\begin{aligned}\mathbb{E}\left[\|u^h(T)\|_{L^2}^2 - \|u^h(0)\|_{L^2}^2\right] &= 2\gamma\mathbb{E}\left[\int_0^T \left\langle Au^h(t,x), u^h(t,x)\right\rangle_{H_0^1} dt\right] \\ &- 2\,\mathbb{E}\left[\int_0^T \left(V'(u^h), u^h(t,x)\right)_{L^2} dt\right].\end{aligned} \tag{5.48}$$

By Lemma 5.4.3,

$$\begin{aligned}
\mathbb{E}\left[\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2\right] &\leq \liminf_{h\to 0} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2 \;-\; \|u^h(0)\|_{L^2}^2\right] \\
&= \liminf_{h\to 0}\; 2\gamma\mathbb{E}\left[\int_0^T \left\langle Au^h(t,x), u^h(t,x)\right\rangle_{H_0^1} dt\right] \\
&- \liminf_{h\to 0}\; 2\,\mathbb{E}\left[\int_0^T \left(V'(u^h), u^h(t,x)\right)\right)_{L^2} dt \right] \\
&= \; 2\mathbb{E}\left[\int_0^T \left\langle \xi, u^h(t,x)\right\rangle_{H_0^1} dt\right] \\
&- \; 2\mathbb{E}\left[\int_0^T \left(\zeta, u^h(t,x)\right)\right)_{L^2} dt \right].
\end{aligned} \tag{5.49}$$

Now we want to use the monotonicity condition to identify the limit $Au = \xi$ and $V'(u) = \zeta$. For this, we need the following lemma:

Lemma 5.4.5. *The monotonicity condition passes to the limit as $h \to 0$.*

Proof of Lemma 5.4.5. By monotonicity of A^h and $V'(u^h)$, for any h

$$\begin{aligned}
\int_0^T \left(A^h u^h - V'(u^h), u^h\right)_{L^2} dt + \int_0^T \left(A^h v - V'(v), v\right)_{L^2} \\
- \left(A^h u^h - V'(u^h), v\right)_{L^2} - \left(A^h v - V'(v), u^h\right)_{L^2} dt \\
\leq 0.
\end{aligned} \tag{5.50}$$

By weak convergence (Prop.5.4.1),

$$\begin{aligned}
\int_0^T \left(A^h u^h, v\right)_{L^2} dt &\rightharpoonup \int_0^T (\xi, v)_{L^2}\, dt \\
\int_0^T \left(V'(v), u^h\right)_{L^2} dt &\rightharpoonup \int_0^T (V'(v), u)_{L^2}\, dt.
\end{aligned} \tag{5.51}$$

By smoothness of the test function v we have

$$\int_0^T \left(A^h v - V'(v), v\right)_{L^2} \rightharpoonup \int_0^T (Av - V'(v), v)_{L^2}\,. \tag{5.52}$$

As we need to make sure that the first term $\int_0^T \left(A^h u^h, u^h\right)_{L^2} dt$ preserves the sign in the limit, i.e. we should verify the following **Claim:**

$$\mathbb{E}\left[\int_0^T \langle \xi - \zeta, u\rangle\; dt\right] \;\leq \liminf_{h\to 0} \mathbb{E}\left[\int_0^T \langle A^h u^h, u^h\rangle\right] dt. \tag{5.53}$$

Proof of Claim: Recall the tested approximative problem (5.28), which reads

$$\begin{aligned} \|u^h(T)\|_{L^2}^2 \;=\; & \|u^h(0)\|_{L^2}^2 \;+\; 2\gamma \int_0^T \left(A^h u^h(t,x), u^h(t,x)\right)_{L^2} dt \\ & -\; 2 \int_0^T \left(V'(u^h(t,x)), u^h(t,x)\right)_{L^2} dt \\ & +\; 2 \int_0^T \left(u^h(t,x), dW^h(t,x)\right)_{L^2} . \end{aligned} \tag{5.54}$$

Take the expectation to get

$$\begin{aligned} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2 - \|u^h(0)\|_{L^2}^2\right] \;=& \; 2\gamma \mathbb{E}\left[\int_0^T \left(A^h u^h(t,x), u^h(t,x)\right)_{L^2} dt\right] \\ & - 2\,\mathbb{E}\left[\int_0^T \left(V'(u^h(t,x)), u^h(t,x)\right)_{L^2} dt\right] . \end{aligned} \tag{5.55}$$

Choosing u as a test function in the weak limit equation (5.47), and taking the expectation, we arrive at

$$\begin{aligned} \mathbb{E}\left[\|u(T)\|^2 - \|u_0\|^2\right] \;=\; & 2\gamma \mathbb{E}\left[\int_0^T \langle \xi, u\rangle_{L^2} dt\right] \\ & - 2\,\mathbb{E}\left[\int_0^T (\zeta, u)_{L^2} dt\right] . \end{aligned} \tag{5.56}$$

As by (5.45)

$$\mathbb{E}\left[\|u(T)\|_{L^2}^2 - \|u_0\|_{L^2}^2\right] \;\le\; \liminf_{h\to 0} \mathbb{E}\left[\|u^h(T)\|_{L^2}^2 - \|u_0^h\|_{L^2}^2\right] \tag{5.57}$$

we get bound (5.56) from above by

$$\gamma \mathbb{E}\left[\int_0^T \langle \xi, u\rangle_{L^2} - (\zeta, u)_{L^2} dt\right] \le \liminf_{h\to 0} \mathbb{E}\left[\int_0^T \left(A^h u^h(t,x), u^h\right)_{L^2} dt\right] , \tag{5.58}$$

which proves the claim.

Consequently, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \langle \xi, u(x)\rangle_{L^2}\, dt - \liminf_{h\to 0}\int_0^T \left(\gamma A^h u^h(t,x), u^h(t,x)\right)_{L^2} dt\right]\\
-\ &\mathbb{E}\left[\int_0^T (\zeta, u(x))_{L^2}\, dt + \liminf_{h\to 0}\int_0^T \left(V'(u^h(t,x)), u^h(t,x)\right)_{L^2} dt\right]\\
&\leq 0,
\end{aligned} \tag{5.59}$$

which means that the monotonicity condition (5.50) passes to the limit and reads

$$\mathbb{E}\left[\int_0^T \langle \xi - \zeta - Av, u(x,t) - v(x,t)\rangle_{L^2}\, dt\right] \leq 0. \tag{5.60}$$

This concludes the proof of the Lemma. □

Continuation of the proof of Theorem 5.4.4 : To conclude that $\xi = Au$ and $\zeta = V'(u)$, we use the hemicontinuity condition: We replace the test function v by another admissible test function w in $L^2\left(\Omega_T \times H_0^1(D)\right)$, more precisely

$$v(x,t) = u(x,t) - \theta w(x,t),$$

where obviously $u, v, w \in L^2\left(\Omega_T \times H_0^1(D)\right)$.
Dividing by θ, (5.60) then reads

$$\mathbb{E}\left[\int_0^T \langle \xi - \zeta - A(u(x,t) - \theta w(x,t)), w(x,t)\rangle_{L^2}\, dt\right] \leq 0 \tag{5.61}$$

for any $w \in L^2\left(\Omega_T \times H_0^1(D)\right)$. By hemicontinuity, the map $\theta \to \langle A(u(x,t) - \theta w(x,t))\rangle$ is continuous from $\mathbb{R} \to \mathbb{R}$. Therefore, it is admissible to pass to the limit $\theta \to 0$ and we reach

$$\mathbb{E}\left[\int_0^T \langle \xi - \zeta - A(u(x,t)), w(x,t)\rangle_{L^2}\, dt\right] \leq 0 \tag{5.62}$$

for any $w \in L^2\left(\Omega_T \times H_0^1(D)\right)$. Since w is arbitrary, the left hand side must vanish, hence $\xi - \zeta = Bu$. Setting now $v = u$ we identify $\zeta = V'(u)$. Plugging this result into (5.60) gives $\xi = Au$.
Consequently, $u \in L^2\left(\Omega\times]0,T[; H_0^1(D)\right) \cap L^2\left(\Omega; L^\infty(0,T,L^2(D)\right)$ is a solution to (5.2). Theorem 3.1. from [101], Chapter 2, gives us furthermore $u \in L^2\left(\Omega; C(0,T,L^2(D)\right)$, so $u(t)$ is a measurable $L^2(D)$-valued stochastic process. This concludes the proof. □

Chapter 6

An interacting particle system with long-range interactions

In this chapter, we describe the behaviour of a particle system with long-range interactions, in which the range of interactions is allowed to depend on the size of the system. We give conditions on the interaction strength under which the scaling limit of the particle system is a well-posed stochastic PDE. As a corollary we obtain that the metastable behaviour of the system is described by the Stochastic Allen-Cahn equation, which has been analyzed by Barret, Bovier and Méléard, Barret and Berglund and Gentz.
This chapter is the extended version of a joint work with A. Bovier [26], which is has not been published yet.

6.1 Introduction

Interacting particle systems model complex phenomena in natural and social sciences, such as traffic flow on highways or pedestrians, opinion dynamics, spread of epidemics or fires, reaction diffusion systems, crystal surface growth, chemotaxis and financial markets. These phenomena involve a large number of interacting components, which are modeled as particles confined to a lattice. Their motion and interaction is governed by local rules, plus some microscopic influences, which is modeled by an independent source of noise. Such noise can either be present in nature or it represents unresolved degrees of freedom.

In this work, we analyse systems of N interaction diffusions on a one-dimensional torus in bistable potentials, where the interaction between particles has range $R \geq 1$. We are interested in the limit as N tends to infinity and R is allowed to diverge with N.

The nearest neighbour case $R = 1$ has been studied in recent years by many authors. Due to the competition between local dynamics and coupling between different sites, a wide range of interesting behaviour was observed and investigated. Before stating our model and the main result, we give a short overview on the history and some results on the nearest-neighbour case, which is very much related to our case:

Without noise, we know that there exist two stable states of the system. In presence of noise, the behaviour of the system is fundamentally different: Arbitrarily small random fluctuations can enable transitions between stable states at large time scales. Whether such transitions are observed will depend on the timescale of interest. The related concepts of phase transition, metastability and metastable timescales have been developed in the context of statistical-mechanics type models, for an overview see the recent book [24] and the references therein. For fixed values of N, this was investigated in [19] and [105]. For large system size N, the behaviour of the nearest-neighbour interaction system is closer to the behaviour of a Ginzburg-Landau partial differential equation with noise, see [45], [106] and [112], for example. More precisely, it was shown in [66] and [67] that after suitable rescaling, the particle system (6.1) converges as N tends to infinity to a stochastic partial differential equation, the stochastic Allen-Cahn equation (see (6.2)). Existence and uniqueness of solutions to this equation in one space dimensions has been proven in [68] via an approximation procedure similar to that in [82].

The question we address here is how fast R can be allowed to grow for the same result to hold, provided we choose an appropriate scaling of the interaction strength. The case $R = N$ corresponds to the mean-field or local mean-field models that has been analysed extensively, see e.g. [40, 57, 58, 93, 83, 95, 96], and where convergence to an SPDE cannot hold. The question we are posing here does not seem to have been addressed in the literature.

Let us mention that the results presented here are limited to one space dimension. In more than one dimension, the SPDE does not have a solution in the classical sense, and one needs to use renormalisation in order to obtain non-trivial results. In two dimensional case this was done by Da Prato and Debussche [38] by using the Wick product in the non-linear term. In dimension three a notion of solution was provided by the framework of Regularity Structures [69], which allows

to treat a large class of non-linear SPDEs. An alternative approach to the equation was provided by [36] using paracontrolled distributions (see [65] as an introduction).
In the recent work [70], Hairer and Matetski developed a systematic approach of spatial discretisations of non-linear SPDEs whose solutions are provided by regularity structures. As an application they proved convergence of the dynamical Φ^4_3 model to the continuous one. The latter result was obtained independently by Zhu and Zhu [120] using paracontrolled calculus.

Setting and statement of the main result We consider a system of N coupled particles on a lattice $\Lambda = \mathbb{Z}/N\mathbb{Z}$. Each particle is subject to a force derived from a bistable potential V and perturbed by Brownian noise. The particle system can be described as a vector-valued function $u^N = (u^N_1, u^N_2, \dots, u^N_N)$ which is a solution of the system of N coupled stochastic differential equations

$$\begin{aligned} du^N_i(t) = \frac{\gamma}{2R^3} \sum_{j=-R}^{R} J_R(j) \left(u^N_{i+j}(t) - u^N_i(t)\right) dt \\ - V'(u^N_i(t))dt + \sqrt{2\sigma} d\widetilde{B}_i(t),\ i \in \Lambda \end{aligned} \tag{6.1}$$

with initial condition $u^N(0) \in \mathbb{R}^N$. Here, $u^N_i(t)$ are the components of the vector $u^N(t) \in \mathbb{R}^N$, $J_R(j) \in \mathbb{R}_+$ are weights, $V(q) = \frac{1}{4}q^4 - \frac{1}{2}q^2$ and $\widetilde{B}_i$ are independent Brownian motions. γ is a constant and $\sqrt{2\sigma}$ the intensity of the noise.
In this model, each particle interacts with all of its neighbours up to distance R. The weights $J_R(j)$ describes the strength of the interaction between two particles at site i and $i + j$.
We look for sufficient conditions on the weights $J_R(j)$ such that, after suitable rescaling, in the limit as $R, N \to \infty$, (6.1) gives rise to a well-posed stochastic PDE,

$$\begin{aligned} \partial_t u(x,t) =& \gamma A u(x,t) - V'(u(x,t)) \ + \sqrt{2\sigma} \frac{\partial^2}{\partial_x \partial_t} W(x,t) \\ & \text{for } (x,t) \in [0,1] \times \mathbb{R}^+ \\ u(0,\cdot) =& u_0, \end{aligned} \tag{6.2}$$

where $A = \frac{1}{2}\Delta$ is the Laplace operator on $[0,1]$ with periodic boundary conditions, $\gamma > 0$ is the diffusion constant, V a double well potential, $\frac{\partial^2}{\partial_x \partial_t} W(x,t)$ denotes space-time white noise and $\sqrt{2\sigma}$ is the intensity of the noise.

After a suitable rescaling of (6.1), which we will present in detail in the next sections, we obtain the following result (see Theorem 6.7.6 and Theorem 6.7.7).

Theorem 6.1.1. *Let $\mathbb{T}_h \subset [0,1]$ denote the equidistant grid on the interval $[0,1]$ with grid size $h = 1/N$. Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u(x,t)$ the solution to (6.2) and $u^h(x,t)$ the piecewise linear function obtained from the solution of the system of SDEs*

$$du^h(ih,t) = \left(\frac{\gamma}{R^3h^2} \sum_{j=-R}^{R} J_R(j) \left(u^h((i+j)h,t) - u^h(ih,t)\right) \right) dt - V'(u^h(ih,t))dt + \sqrt{\frac{2\sigma}{h}} dB_i(t),\ i \in \mathbb{T}_h, \tag{6.3}$$

with $J_R(j) = J\left(\frac{j}{R}\right)$, where J is positive and satisfies $\int J(x)x^2\,dx = 1$. If $R \sim h^{-\zeta}$ with $\zeta < \frac{1}{2}$, then
i) for all times $T > 0$, and all $p > 1$, $u^h \longrightarrow u$ in $L^p\left(\Omega, C([0,1]\times[0,T])\right)$.
ii) for all times $T > 0$, there exists an almost surely finite random variable $\mathbb{X}$ such that

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \ \leq\ \mathbb{X}h^\eta$$

for $0 < \eta < \frac{1}{2} - \delta$.

6.2 Properties of the discrete operator

Notation and rescaling. We rescale the unit lattice $\Lambda = \mathbb{Z}/N\mathbb{Z}$ by $h = \frac{1}{N}$ to arrive at the uniform grid $\mathbb{T}_h = \{0, h, \ldots, Nh\}$ where we identify $0 = 1 = Nh$. $\mathbb{T}_h$ is then a discretization of the interval $[0,1]$ in equidistant nodes. We call h the grid size and will sometimes refer to ih as the "node i".
Moreover, we rescale the coupling constant $\tilde{\gamma}$ by h^{-1} and the potential term by h. Then we accelerate time by a factor $\frac{1}{h}$, i.e. we set $\widetilde{X}(t) = X(t/h)$, which gives us another extra h^{-1} on the coupling constant and cancels out the previous changes in the scaling of the potential. Moreover, this acceleration of time gives us a different sequence of independent Brownian motions, which we call $B_i(t)$. The real-valued stochastic process $\widetilde{X}_i^h(t)$ can then be identified with the real-valued function $u_i(t,\omega)$ of nodal values at the node i. The resulting rescaled

system of SDEs reads

$$du_i(t) = \left(\frac{\gamma}{R^3h^2} \sum_{j=-R}^{R} J_R(j) \left(u_{i+j}(t) - u_i(t)\right) \right) dt$$
$$- V'(u_i(t))dt + \sqrt{\frac{2\sigma}{h}} dB_i(t),$$

for all $i \in \mathbb{T}_h$. Note that $u_i(t)$ is defined only at one specific node i. Via $u_i(t) := u^h(ih,t)$, the vector-valued function of nodal values $u^h(t) = (u_1(t), u_2(t), \ldots, u_N(t))$ on the grid $\mathbb{T}_h$ can be identified as a continuous, piecewise linear function on $u^h(x,t) : D \times \mathbb{R}^+ \to \mathbb{R}$.
Note furthermore that we can relate the rescaled Brownian noise to space-time white noise via

$$\sqrt{h} B_i(t) = \int_{(i-1)h}^{ih} W(x,t)\, dx. \tag{6.4}$$

We can rewrite (6.3) in integral form as

$$\begin{aligned} u^h(x,t) \;=\; & \int_0^1 g_t^h(x,y) u_0^h(y)\, dy \\ & - \int_0^t \int_0^1 g_{t-s}^h(x,y) V'((u^h(y,s))\, dy ds \\ & + \sqrt{2\sigma} \int_0^t \int_0^1 g_{t-s}^h(x,y)\, W(dy,ds) \end{aligned} \tag{6.5}$$

where $g_t^h(x,y)$ is the semigroup associated with the discrete operator $-\gamma A_R^h$ defined in (6.7). $g_t^h(x,y)$ is defined on $\mathbb{R}^+ \times [0,1] \times [0,1]$ in a piecewise linear fashion, see (6.29).

Central difference operator and weights. We interpret the collection of weights $J_R(j)$ as a weight function $J_R : \mathbb{Z} \to \mathbb{R}_+$, $J_R(j) = J\left(\frac{j}{R}\right)$ for some positive function J satisfying

$$\int_{\mathbb{R}} x^2 J(x) dx \;=\; 1 \tag{6.6}$$

Typical examples are $J_R(j) = c \exp(-j/R)$ or $J_R(j) = c\mathbf{1}_{|j|\leq R}$. The weights $J_R(j)$ are the entries of the j-th subdiagonal of the band matrix A_R^h, where R indicates the width of the stencil:

$$A_R^h u_i = \frac{J_R(R)u_{i+R} + \ldots + J_R(1)u_{i+1} - J_R(0)u_i + J_R(-1)u_{i-1} \ldots J_R(-R)u_{i-R}}{R^3h^2} \tag{6.7}$$

Note that the weights $J_R(j)$ are fixed and do not change with time, so the central difference operator $-A_R^h$ is time-independent. Moreover, $-A_R^h$ is a positive definite matrix, as, by construction from the model (6.3), the values of the weight function satisfy the diagonal dominance relation $J_R(0) = 2\sum_{j\geq 1} J_R(j) \;=\; 2\sum_{j=1}^{R} J_R(j)$ where $J_R(0)$ is the weight attributed to the reference site i.

Boundedness of the inverse. The big difference between the particle system with long-range interaction and a particle system with nearest-neighbour interaction is that the interaction length R actually tends to infinity as the number of particles go to infinity.
We have modeled our particle system as a discretization in the space variable of a continuous limit, which means that instead of discussing the limit as the number of particles N go to infinity, we actually consider the limit $h \to 0$ of a semidiscrete finite difference scheme (with $h = \frac{1}{N}$). As we consider the simultaneous limit of both variables h and R, it is convenient to rewrite R in terms of h, so we define $R = h^{-\zeta}$ with $0 < \zeta < 1$. In the next lemma we derive the admissible values of ζ such that $(-A_R^h)^{-1}$ is a bounded operator:

Lemma 6.2.1. *Let*

$$\gamma A_R^h u_i = \frac{\gamma}{R^3 h^2} \sum_{j=-R}^{R} J_R(j) \left(u_{i+j}(t) - u_i(t)\right) \tag{6.8}$$

with $R \sim h^{-\zeta}$, and $J_R(j)$ satisfying (6.6). *Let the eigenvalues of* (6.8) *with periodic boundary conditions be denoted by*
$\lambda_k^h = \frac{4\gamma}{h^2 R^3} \sum_{j=1}^{R} J_R(j) \sin^2\left(\frac{\pi}{2} khj\right)$.
Then, for $\zeta < \frac{1}{2}$

$$\sum_{k=1}^{1/h} \left(\lambda_k^h\right)^{-1} \;\leq\; \sum_{k=1}^{h^{\zeta-1}} \frac{1}{ck^2} + o(h^{1-2\zeta}) \;<\; \infty. \tag{6.9}$$

Proof of Lemma 6.2.1. Consider without loss of generality $J_R(j) = c\mathbf{1}_{|j|\leq R}$ with some constant c chosen such that

$$\frac{c}{R^3} \sum_{j=-R}^{R} J_R(j) j^2 = 1, \tag{6.10}$$

which is a discrete version of 6.6, the moment condition on J. We need to show that

$$\sum_{k=1}^{1/h} \left(\lambda_k^h\right)^{-1} \;\leq\; \sum_{k=1}^{1/h} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2} khj\right)} \;<\infty. \tag{6.11}$$

We split the sum in $k \leq h^{\zeta-1}$ and $k > h^{\zeta-1}$. As for $k \leq h^{\zeta-1}$ we have $khj \leq 1$, the increment of the sine squared stays inside the regime $\left[0, \frac{\pi}{2}\right]$. In this regime, we use the fact that $\sin^2(x) \geq \frac{4x^2}{\pi^2}$ to estimate

$$\sum_{k=1}^{h^{\zeta-1}} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} \leq \sum_{k=1}^{h^{\zeta-1}} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} k^2h^2j^2} \leq \sum_{k=1}^{h^{\zeta-1}} \frac{1}{k^2} \tag{6.12}$$

which gives the first term in (6.9).
For $k > h^{\zeta-1}$, note that for any j there exists a $k(j)$ such that $khj = 1$, for which we have $\sin^2\left(\frac{\pi}{2}khj\right) = 1$. Therefore, the denominator $\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)$ can be bounded from below by this element, which has the value $\sin^2(1) = 1$. We split furthermore

$$\begin{aligned}\sum_{k>h^{\zeta-1}}^{1/h} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} &= \sum_{k>h^{\zeta-1}}^{1/(2h)} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} \\ &\quad + \sum_{k>1/(2h)}^{1/h} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)}.\end{aligned} \tag{6.13}$$

Note that in the regime $\frac{h^{-1}}{2} \leq k \leq h^{-1}$, the increment has the bounds

$$\frac{1}{2} \leq \frac{j}{2} \leq khj \leq j \leq \frac{1}{2}h^{-\zeta} \tag{6.14}$$

so by choosing $j = 1$ we immediately get that the denominator is bounded from below by $\sin^2\left(\frac{1}{2}\right)$.
We improve this bound in the following way:
First, extend for convenience the sum in the denominator to $j = 0$. As $\sin^2(0) = 0$, this does not change the value in the sum. However, it simplifies (6.14) to $0 \leq khj \leq \frac{1}{2}h^{-\zeta}$, i.e. there are $\frac{1}{2} \cdot \frac{1}{2}h^{-\zeta}$ many terms in the denominator which are bounded from below by $\sin^2\left(\frac{1}{2}\right)$. As there are $\frac{h^{-1}}{2}$ many terms in the k-sum, we get in total

$$\sum_{k>1/(2h)}^{1/h} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} \leq h^{-3\zeta+2} \cdot h^{-1} \cdot h^{-\zeta} = o(h^{-2\zeta+1}). \tag{6.15}$$

For the remaining regime $\tilde{\mathbb{I}} := \left\{h^{\zeta-1} < k < \frac{h^{-1}}{2}\right\}$, we note first that for variable k we have, due to $j \leq h^{-\zeta}$, $jhk \leq k \cdot h^{1-\zeta}$, in other words, with $x := jhk$ we have $x \in [0, k \cdot h^{1-\zeta}]$ and as $k > h^{\zeta-1}$ in $\mathbb{I}$, we

have $[0,1] \subset [0, k \cdot h^{1-\zeta}]$, so we know a priori that $\sin^2\left(\frac{\pi}{2}x\right)$ will be approximately 1 at least once in this interval.
As the sum over k is divided into N pieces, the distance between the evaluation points of $\sin^2$ as a function in the k variable is kh and $kh \leq \frac{1}{2}$ in $\tilde{\mathbb{I}}$.
Choose some $m > 0$. We split the the interval $\mathbb{I} := \left[h^{\zeta-1}, h^{-1}\right]$ into m parts of the form $\mathbb{I}_l := \left[h^{\frac{(m-l+1)\zeta}{m}-1}, h^{\frac{(m-l)\zeta}{m}-1}\right]$. In particular, the first interval has the form $\mathbb{I}_1 := \left[h^{\zeta-1}, h^{\frac{(m-1)\zeta}{m}-1}\right]$ and the last interval has the form $\mathbb{I}_m := \left[h^{\frac{(m-1)\zeta}{m}-1}, h^{-1}\right]$.
First interval: For h sufficiently small, we have that $kh \geq 1$ and therefore $[0,1] \subset [0, k \cdot h^{1-\zeta}]$. Moreover, there exists j_0 such that khj is an odd integer, which means that $\sin^2\left(\frac{\pi}{2}khj_0\right) = 1$. As $|\mathbb{I}_1| = \lfloor h^{\zeta-\zeta/m-1} - h^{\zeta-1}\rfloor$, we get

$$\begin{aligned}\sum_{k\in\mathbb{I}_1} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} \;&\leq\; h^{-3\zeta+2} \cdot \left(h^{\zeta-\zeta/m-1} - h^{\zeta-1}\right) \\ &\leq\; h^{-(2+\frac{1}{m})\zeta+1} - h^{-2\zeta+1} \;=\; o(h^{1-(2+\frac{1}{m})\zeta})\end{aligned} \tag{6.16}$$

which is bounded as long as $1-(2+\frac{1}{m})\zeta > 0$, i.e. as long as $\zeta < \frac{1}{2}$.
Middle intervals: For $x := khj$ as before and extending to $j = 0$ as before, we get that $x \in \left[0, h^{\frac{-l}{m}\zeta}\right]$, so the denominator gives $h^{\frac{-l}{m}\zeta}$ many contributions of size ≈ 1:

$$\begin{aligned}\sum_{k\in\mathbb{I}_l} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} \;&\leq\; h^{-(2+\frac{l}{m}-\frac{l}{m})\zeta+1} - h^{-(2+\frac{l+1}{m}-\frac{l}{m})\zeta+1} \\ &\leq\; h^{-2\zeta+1} - h^{-(2+\frac{1}{m})\zeta+1}\end{aligned} \tag{6.17}$$

which gives exactly the same bound as the first interval.
Note that the last interval is in general still bigger than the interval considered in (6.15), in fact it contains the interval $\left[\frac{1}{2h}, \frac{1}{h}\right]$. But we know that $\sin^2\left(\frac{\pi}{2}x\right)$ is two-periodic, $\sin^2\left(\frac{\pi}{2}x\right) \geq \epsilon_0$ for all odd integers x contained in the interval $\mathbb{I}_m$.
There are $\frac{1}{2}|\mathbb{I}_m|$ many of these x. Therefore, we conclude the same bound as in (6.17).

Altogether, this gives

$$\sum_{k>h^{\zeta-1}}^{1/h} \frac{h^{-3\zeta+2}}{\sum_{j=1}^{h^{-\zeta}} \sin^2\left(\frac{\pi}{2}khj\right)} \;\leq\; \left(h^{-2\zeta+2} \cdot h^{-1}\right) \;=\; o(h^{-2\zeta+1}) \tag{6.18}$$

which concludes the proof. □

A direct consequence of Lemma 6.2.1 is

$$\frac{c}{R^3}\sum_{j=1}^{R} J_R(j) j^4 = o(h^{-2\zeta}) \tag{6.19}$$

which we use in the forthcoming Lemma of convergence of the eigenvalues.

Convergence of eigenvalues and eigenvectors. In the following, we state some useful facts on the eigenvalues and eigenvectors of the long-range discrete operator $-\gamma A_R^h$.
Notation: From now on, when writing $-\gamma A_R^h$ we mean the discrete symmetric stencil with appropriate choices of coefficient and scaling as stated in (6.8).
Due to the symmetry of $-\gamma A_R^h$, its eigenvalues under periodic boundary conditions can be computed as

$$\lambda_k^h = \frac{4\gamma}{h^2 R^3}\sum_{j=1}^{R} J_R(j)\sin^2\left(\frac{\pi}{2}khj\right). \tag{6.20}$$

Property (6.10) gives immediately the upper bound

$$\lambda_k^h \leq \gamma\pi^2 k^2 = \lambda_k \tag{6.21}$$

and as for $0 \leq x \leq \frac{\pi}{2}$, $\sin^2(x) \geq \frac{4x^2}{\pi^2}$, the following lower bound holds

$$\lambda_k^h \geq \frac{4\gamma k^2}{R^3}\sum_{j=1}^{R} J_R(j) j^2 \overset{(6.10)}{=} 4\gamma k^2. \tag{6.22}$$

We will use this inequality frequently for estimates on the discrete semigroup.
By Lemma 6.2.1, we can conclude convergence of the eigenvalues:

Lemma 6.2.2. *Let λ_k be the eigenvalues of A on $[0,1]$ and λ_k^h given by (6.20) the eigenvalues of the the long-range discrete operator $-\gamma A_R^h$. Let $R \sim h^{-\zeta}$ and $J_R(j)$ as in Lemma 6.2.1. Then we have*

$$\lambda_k - \lambda_k^h \longrightarrow 0 \quad \text{as } h \to 0 \tag{6.23}$$

with convergence rate smaller than $h^{2-2\zeta}$.

Proof of Lemma 6.2.2. We use the fact that for $0 \leq x \leq \frac{\pi}{2}$, $\sin^2(x) \geq x^2(1 - \frac{x^2}{3})$:

$$\begin{aligned}\lambda_k^h &\geq \frac{\gamma\pi^2 k^2}{R^3} \sum_{j=1}^{R} \left\{ J_R(j) j^2 \left(1 - \frac{1}{12} k^2 h^2 \pi^2 j^2 \right) \right\} \\ &\geq c\gamma\pi^2 k^2 \underbrace{\frac{1}{R^3} \sum_{j=1}^{R} J_R(j) j^2}_{\overset{(6.10)}{=} 1} - \frac{\gamma}{12} k^4 \pi^4 h^2 \underbrace{\frac{1}{R^3} \sum_{j=1}^{R} J_R(j) j^4}_{\overset{(6.19)}{=} h^{-2\zeta}} . \end{aligned} \tag{6.24}$$

Therefore,

$$\lambda_k - \lambda_k^h \leq \frac{\gamma}{12} k^4 \pi^4 \cdot h^{2-2\zeta} = c(\gamma) h^{2-2\zeta}. \tag{6.25}$$

Note that the rate of convergence depends on ζ. Since we have chosen $\zeta < \frac{1}{2}$, the rate $h^{2-2\zeta}$ is at most $h^{1+\epsilon}$. □

Eigenvectors. The m-th entry of the eigenvector v_k^h (the eigenvector of $-\gamma A_R^h$ associated with the eigenvalue λ_k^h) reads

$$v_k^h(m) = \sin(\pi k m h). \tag{6.26}$$

Obviously, we get the following result:

Lemma 6.2.3. *Given $v_k = \sin(\pi k x)$ the eigenfunction of the Laplace operator with periodic boundary conditions on $[0, 1]$ assiociated to the eigenvalue λ_k and $v_k^h = (v_k^h(1), \ldots v_k^h(N))$ with $v_k^h(i)$ as in (6.26) the eigenvectors of $-\gamma A_R^h$. Given $x \in [h(m - \frac{1}{2}), h(m + \frac{1}{2})[$. Then, as $h \to 0$,*

$$|v_k(x) - v_k^h(m)| \longrightarrow 0 \qquad \text{as } h \to 0. \tag{6.27}$$

Consistency of difference operators. Assuming sufficient regularity of the solution up to the boundary of the domain, to ensure convergence of a finite difference operator, we need that it is *consistent*, which means a vanishing local error as the grid size goes to zero. The order of consistency tells us about the rate of convergence of a difference stencil to a continuous operator. It is derived using the Taylor formula and comparing the coefficients. This approach leads to high regularity restrictions such as $u \in C^{m+2}$, see for example [115] for details.

Exploiting cancellation effects given by the symmetry of the stencil and the equidistant grid, we get for A_R^h as in (6.8) with initial data

$u_0 \in C^4([0,1])$

$$\sup_{y\in[0,1]} \left|A_R^h u(y) - u_{xx}(y)\right| \le O(h^2). \tag{6.28}$$

6.3 Convergence of the discrete semigroup

Let $-\gamma A_R^h$ as in in (6.8). Denote by λ_k^h the eigenvalues of $-\gamma A_R^h$ stated in (6.20) and by v_k^h the piecewise linear functions on the grid realized by the eigenvectors of $-\gamma A_R^h$, see (6.26). The discrete semigroup associated to $-\gamma A_R^h$ reads

$$g_t^h(x,y) = \sum_{k=1}^{1/h} e^{-t\lambda_k^h} v_k^h(x) v_k^h(y). \tag{6.29}$$

Due to Lemma 6.2.2 and 6.2.3, we can already prove uniform convergence of g^h to the heat kernel g on $[t_0,\infty)\times[0,1]^2$:

Proposition 6.3.1. *Let $g_t(x,y)$ the heat kernel (see* (6.83)*) and $g_t^h(x,y)$ as in* (6.29) *with eigenvalues as stated in* (6.20) *and eigenvectors* (6.26)*. For all $t_0 > 0$ and $\zeta < \frac{1}{2}$, there exists a constant $c(\gamma,t_0)$ such that for all $(t,x,y)\in[t_0,\infty)\times[0,1]^2$*

$$|g_t^h(x,y) - g_t(x,y)| \le c(\gamma,t_0) h^{2-2\zeta}. \tag{6.30}$$

Proof of Proposition 6.3.1 . **Step 1: estimates for fixed** k: We first look at the difference $g^h(x,y) - g(x,y)$ for fixed k.

$$\left|e^{-\lambda_k^h t} v_k^h(x) v_k^h(y) - e^{-\lambda_k t} v_k(x) v_k(y))\right| \le (I) + (II) + (III) \tag{6.31}$$

where

$$\begin{aligned}
(I) &= \left|\left(e^{-\lambda_k^h t} - e^{-\lambda_k t}\right) v_k^h(x) v_k^h(y)\right| \;\le\; 2e^{-\lambda_k^h t}|1 - e^{-t(\lambda_k - \lambda_k^h)}| \\
(II) &= \left|e^{-\lambda_k t} v_k^h(y)\left(v_k^h(x) - v_k(x)\right)\right| \;\le\; e^{-\lambda_k t}\cdot\sqrt{2}\cdot\left|v_k^h(x) - v_k(x)\right| \\
(III) &= \left|e^{-\lambda_k t} v_k(x)\left(v_k^h(y) - v_k(y)\right)\right| \;\le\; e^{-\lambda_k t}\cdot\sqrt{2}\cdot\left|v_k^h(y) - v_k(y)\right|.
\end{aligned} \tag{6.32}$$

Indeed, multiplication with $e^{-\lambda_k^h t}\cdot e^{\lambda_k^h t}$ from the left gives

$$\begin{aligned}
&\Big|e^{-\lambda_k^h t}\cdot\big(\underbrace{e^{\lambda_k^h t} e^{-\lambda_k^h t}}_{=1} - \underbrace{e^{\lambda_k^h t} e^{-\lambda_k t}}_{=e^{\lambda_k^h t - \lambda_k t}}\big) v_k^h(x) v_k^h(y)\Big| \\
&\qquad\le \left|e^{-\lambda_k^h t}\left(1 - e^{-t(\lambda_k - \lambda_k^h)}\right) v_k^h(x) v_k^h(y)\right| \\
&\qquad\le 2e^{-\lambda_k^h t}|1 - e^{-t(\lambda_k - \lambda_k^h)}| \\
&\qquad\overset{(6.24)}{\le} 2e^{-\lambda_k^h t}|1 - e^{-th^{2-2\zeta}}|,
\end{aligned} \tag{6.33}$$

which justifies (I). Moreover,

$$\begin{aligned}\left|e^{-\lambda_k t} v_k(x)\big(v_k^h(y)-v_k(y)\big)\right| &\leq e^{-\lambda_k t}\cdot\left|v_k(x)(v_k^h(y)-v_k(y))\right|\\ &\leq e^{-\lambda_k t}\cdot\sqrt{2}\cdot\left|v_k^h(y)-v_k(y)\right| \\ &\overset{(6.27)}{\leq} \sqrt{2}\pi k h\cdot e^{-\lambda_k t}\end{aligned} \tag{6.34}$$

which justifies (II) and (III).

Step 2: convergence of the semigroup Now we sum over all terms, using the above estimates:

$$\begin{aligned}|g_t^h(x,y)-g_t(x,y)| \leq &\sum_{k=1}^{1/h}\underbrace{\left|e^{-\lambda_k^h t}v_k^h(x)v_k^h(y)-e^{-\lambda_k t}v_k(x)v_k(y))\right|}_{(*)}\\ &+\sum_{k>1/h}^{\infty} e^{-\lambda_k t}v_k(x)v_k(y).\end{aligned} \tag{6.35}$$

First, we estimate $(*)$:

$$\begin{aligned}(*) &= \sum_{k=1}^{1/h}\left|e^{-\lambda_k^h t}v_k^h(x)v_k^h(y)-e^{-\lambda_k t}v_k(x)v_k(y))\right|\\ &\overset{(6.31)}{\leq} \sum_{k=1}^{1/h}(I)+(II)+(III)\\ &\overset{(6.24)}{\leq} 2\sum_{k=1}^{1/h}\left(e^{-\lambda_k^h t}\left|1-e^{-t\gamma k^4\frac{\pi^4}{12}h^{2-2\zeta}}\right|+\sqrt{2}\pi k h\cdot e^{-\lambda_k t}\right)\\ &\overset{(6.22)}{\leq} 2\sum_{k=1}^{1/h}\left(e^{-4\gamma k^2 t}\left|1-e^{-t\gamma k^4\frac{\pi^4}{12}h^{2-2\zeta}}\right|+\sqrt{2}\pi k h\cdot e^{-\gamma\pi^2 k^2 t}\right)\\ &\leq 2\sum_{k=1}^{1/h} e^{-4\gamma\pi k^2 t}\left|1-e^{-t\gamma k^4\frac{\pi^4}{12}h^{2-2\zeta}}\right| \;+\; 2\sqrt{2}\pi\sum_{k=1}^{1/h} kh\cdot e^{-\gamma\pi^2 k^2 t}.\end{aligned} \tag{6.36}$$

The second sum of the RHS of (6.36) is converging to zero as with (6.46)

$$2\sum_{k=1}^{1/h}\sqrt{2}\pi k h\cdot e^{-\gamma\pi^2 k^2 t} \leq h\cdot\frac{\sqrt{2}}{\pi\gamma t}. \tag{6.37}$$

To estimate the first term, note that we can estimate with (6.42)

$$1 - \exp\left(-t\gamma k^4 \frac{\pi^4}{12} h^{2-2\zeta}\right) \le \min\left\{1, t\gamma k^4 \frac{\pi^4}{12} h^{2-2\zeta}\right\}. \tag{6.38}$$

Therefore, the first term of the RHS of (6.36) can be estimated as

$$\begin{aligned} 2\sum_{k=1}^{1/h} e^{-4\gamma\pi k^2 t} \left|1 - e^{-t\gamma k^4 \frac{\pi^4}{12} h^{2-2\zeta}}\right| &= \frac{\pi^4}{6}\gamma t h^{2-2\zeta} \sum_{k=1}^{1/h} k^4 \exp(-4\gamma\pi k^2 t) \\ &= \frac{\pi^4}{6}\gamma t h^{2-2\zeta} \frac{1}{2(4\pi\gamma t)^{5/2}} \Gamma\left(\frac{5}{2}\right) \\ &= h^{2-2\zeta} \frac{\pi^{\frac{3}{2}}}{3 \cdot 2^8 (\gamma t)^{3/2}} \underbrace{\Gamma\left(\frac{5}{2}\right)}_{=\frac{3}{4}\sqrt{\pi}} \\ &= h^{2-2\zeta} \underbrace{\frac{\pi^2}{64 \cdot (\gamma t)^{3/2}}}_{=c(\gamma,t)} \\ &\le c(\gamma, t) h^{2-2\zeta}. \end{aligned} \tag{6.39}$$

Last, we estimate

$$\begin{aligned} \sum_{k>1/h}^{\infty} e^{-\lambda_k t} v_k(x) v_k(y) &\le 2\int_{1/h}^{\infty} e^{-\gamma\pi^2 k^2 t} dk \\ &\le 2h^2 \int_0^{\infty} k^2 e^{-\gamma\pi^2 k^2 t} dk \\ &\le \frac{h^2}{(\gamma\pi^2 t)^{\frac{3}{2}}} \Gamma\left(\frac{3}{2}\right). \end{aligned} \tag{6.40}$$

We insert (6.39), (6.37) and (6.40) in (6.35) to get finally

$$\sum_{k=1}^{1/h} \left| e^{-\lambda_k^h t} v_k^h(x) v_k^h(y) - e^{-\lambda_k t} v_k(x) v_k(y)) \right| \le c(\gamma, t)\left(h^{2-2\zeta} + h + h^2\right). \tag{6.41}$$

The constant $c(\gamma, t)$ is biggest when t is smallest, which is the case when $t = t_0$. This explains the appearance of $c(\gamma, t_0)$ in the statement of this proposition, whose proof is now concluded. □

6.4 Estimates on the stochastic integral

In the following section, we derive the necessary a priori estimates on the discrete solution. For this, we state some useful facts for the convenience of the reader. We will use them frequently in the whole section.

Useful estimates Using

$$1 - \exp(-x) \leq \min\{1, x\} \tag{6.42}$$

one can show the following useful estimates:

$$\sum_{k=1}^{1/h} \frac{1}{k^2} \left(1 - \exp(-ck^2)\right) \ \leq \ 3(\sqrt{c} \wedge 1) \tag{6.43}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \left(1 - \exp(-alk^2)\right)^2 \leq \frac{1}{k^2} \left[1 \wedge alk^2\right]^2 \leq 3(\sqrt{al} \wedge 1). \tag{6.44}$$

Our favourite choices of the variables a and l are $a = (t'-t)$ and $l \sim \frac{\lambda_k^h}{k^2}$) Moreover, we sometimes employ the following integral identities:

$$\sum_{k=1}^{1/h} \exp(-c \cdot 8\pi k^2) \leq \int_0^\infty e^{-c \cdot 8k^2\pi} dk = \frac{1}{4\sqrt{2}\sqrt{c}} \tag{6.45}$$

$$\sum_{k=1}^{1/h} k \cdot e^{-\gamma\pi^2 k^2 t} \leq \int_0^\infty k \cdot e^{-\gamma t \pi^2 k^2} = \frac{1}{2\pi^2 \gamma t} \tag{6.46}$$

and

$$\int_0^\infty t^2 e^{-ct} dt = \frac{2}{c^3}. \tag{6.47}$$

6.4.1 Estimates on the discrete semigroup

In the following, we prove some estimates on the discrete semigroup (6.29). We start with an estimate on the L^2 norm of (6.29) on the torus:

Lemma 6.4.1. *Let* $g_t^h(x,y) = \sum_{k=1}^{1/h} \exp(-t\lambda_k^h) v_k^h(x) v_k^h(y)$ *be the discrete semigroup defined in* (6.29) *with* λ_k^h *the eigenvalues of* $-\gamma A_R^h$

with periodic boundary conditions and by v_k^h *the piecewise linear functions on the grid realized by the eigenvectors of* $-\gamma A_R^h$.
Then there exists $c = c(\gamma, t)$ *such that for all* $x \in [0,1]$ *and all* $t \in [0,T]$ *the following estimates hold:*

$$\int_0^1 g_t^h(x,y)^2 \, dy \;\le\; c(\gamma,t) \tag{6.48}$$

$$\int_0^1 \int_0^1 g_t^h(x,y)^2 \, dy \, dx \;\le\; c(\gamma,t) \tag{6.49}$$

$$\int_0^t \int_0^1 g_s^h(x,y)^2 \, dy \, ds \;\le\; c(\gamma,t) \tag{6.50}$$

$$\int_0^t \int_0^1 \int_0^1 g_s^h(x,y)^2 \, dy \, dx \, ds \;\le\; c(\gamma,t). \tag{6.51}$$

Proof of Lemma 6.4.1. First inequality: To derive (6.48), we estimate

$$\begin{aligned}
\int_0^1 g_t^h(x,y)^2 dy &= \int_0^1 \left(\sum_{k=1}^{1/h} \exp(-t\lambda_k^h) v_k^h(x) v_k^h(y) \right) \times \\
&\qquad \left(\sum_{k=1}^{1/h} \exp(-t\lambda_k^h) v_k^h(x) v_k^h(y) \right) dy \\
&= \int_0^1 \sum_{k=1}^{1/h} \left(\exp(-t\lambda_k^h) v_k^h(x) v_k^h(y) \right)^2 \, dy \\
&\overset{v_k^h(x) \perp v_k^h(y)}{=} \sum_{k=1}^{1/h} \exp(-2t\lambda_k^h) v_k^h(x)^2 \\
&= 2 \sum_{k=1}^{1/h} \exp(-2t\lambda_k^h) \\
&\overset{(6.22)}{\le} 2 \sum_{k=1}^{1/h} \exp(-t \cdot 8\gamma k^2) \\
&\overset{(6.45)}{\le} \frac{\sqrt{\pi}}{2\sqrt{2}\sqrt{\gamma t}} \;\le\; c(\gamma,t).
\end{aligned} \tag{6.52}$$

The second inequality (6.49) follows from the first by integration, which does not change the bound.

Third inequality: To derive (6.50), we estimate

$$\begin{aligned}\int_0^t \int_0^1 g_s^h(x,y)^2 dyds &= \int_0^t \sum_{k=1}^{1/h} \exp(-2s\lambda_k^h) v_k^h(x)^2 ds \\ &= \sum_{k=1}^{1/h} v_k^h(x)^2 \frac{1}{2\lambda_k^h} \left(1 - \exp(-2t\lambda_k^h)\right) \\ &\overset{(6.22)}{\leq} \frac{1}{8\gamma} \sum_{k=1}^{1/h} \frac{1}{k^2} \left(1 - \exp(-t \cdot 8\gamma k^2)\right) \\ &\overset{(6.43)}{\leq} \frac{3}{8\gamma} \min\left\{\sqrt{8\gamma t}, 1\right\} \leq \frac{3}{\sqrt{8\gamma}} \min\left\{\sqrt{t}, 1\right\} \\ &\leq c(\gamma, t).\end{aligned} \tag{6.53}$$

The fourth inequality (6.51) follows from the first by integration, which does not change the bound. □

6.4.2 Regularity of the discrete semigroup

The next lemma provides estimates on the discrete semigroup as a function on time and space. It is used in Lemma (6.4.3) on the regularity of the stochastic integral.

Lemma 6.4.2. *Given the discrete semigroup* (6.29) *and let* $(x,t), (x',t') \in D \times [0,T]$ *with* $t' > t$*. Then we have the following estimates:*

$$\int_0^t \int_0^1 \left|g_{t-s}^h(x,y) - g_{t-s}^h(x',y)\right|^2 dyds \;\leq\; c(\gamma)|x - x'| \tag{6.54}$$

and in time

$$\int_0^t \int_0^1 \left|g_{t-s}^h(x,y) - g_{t'-s}^h(x,y)\right|^2 dyds \;\leq\; c(\gamma)\sqrt{|t - t'|}. \tag{6.55}$$

Proof of Lemma 6.4.2. Part 1: Proof of (6.54) Take two grid points $x = ih$ and $x' = mh$. As $|v_k^h(y)|^2 = 2$ and $\int_0^t \exp(-2\lambda_k^h(t-s)) \leq \frac{1}{2\lambda_k^h}$, we get

$$\begin{aligned}
&\int_0^t \int_0^1 \left|g^h_{t-s}(x,y) - g^h_{t-s}(x',y)\right|^2 dy\, ds \\
&\qquad \le \sum_{k=1}^{1/h} \int_0^t \exp(-2(t-s)\lambda^h_k)|v^h_k(x) - v^h_k(x')|^2\, ds \\
&\qquad \overset{(6.42)}{\le} \sum_{k=1}^{1/h} \frac{1}{2\lambda^h_k}|v^h_k(x) - v^h_k(x')|^2 \\
&\qquad \overset{(6.22)}{\le} \sum_{k=1}^{1/h} \frac{1}{8\gamma k^2}\pi k|x-x'|^2 \\
&\qquad = c(\gamma)|x-x'|.
\end{aligned} \tag{6.56}$$

Part 2: Proof of (6.55) By orthogonality of the eigenvectors and the fact that $v^h_k(x)^2 = 2$, we can estimate

$$\begin{aligned}
&\int_0^t \int_0^1 \left|g^h_{t-s}(x,y) - g^h_{t'-s}(x,y)\right|^2 dy \\
&\qquad \le \sum_{k=1}^{1/h} \int_0^t \left|\exp(-(t-s)\lambda^h_k) - \exp(-(t'-s)\lambda^h_k)\right|^2 v^h_k(x)^2\, ds \\
&\qquad = 2\sum_{k=1}^{1/h} \int_0^t \left|\exp(-(t-s)\lambda^h_k) - \exp(-(t'-s)\lambda^h_k)\right|^2 ds.
\end{aligned} \tag{6.57}$$

Left-multiplication of a factor equal to one leads to

$$\begin{aligned}
&\int_0^t \int_0^1 \left|g^h_{t-s}(x,y) - g^h_{t'-s}(x,y)\right|^2 dy \\
&\qquad \le 2\sum_{k=1}^{1/h} \int_0^t \exp(-2(t-s)\lambda^h_k)\left|1 - \exp(-(t'-t)\lambda^h_k)\right|^2 ds \\
&\qquad \le 2\sum_{k=1}^{1/h} \frac{1}{2\lambda^h_k}\left|1 - \exp(-(t'-t)\lambda^h_k)\right|^2 \\
&\qquad \overset{(6.22)}{\le} 2\sum_{k=1}^{1/h} \frac{1}{8\gamma k^2}\left|1 - \exp(-(t'-t)\lambda^h_k)\right|^2 \\
&\qquad \overset{(6.42),(6.21)}{\le} c(\gamma)\sqrt{|t-t'|}.
\end{aligned} \tag{6.58}$$

□

6.4.3 Regularity of the discrete stochastic integral

Lemma 6.4.3. *Given the discrete semigroup* (6.29) *and a sequence* u^h *of random variables which satisfy*

$$\sup_h \sup_{t\in[0,T]} \sup_{x\in[0,1]} \mathbb{E}[|u^h(x,t)|^p] \leq C \tag{6.59}$$

Define the stochastic integral

$$S(x,t) = \int_0^t \int_0^1 g^h_{t-s}(x,y) u^h(y,s)\, W(dy,ds) \tag{6.60}$$

Then we have for $1 \leq p < \infty$ *and* $T > 0$

$$\mathbb{E}\Big[\Big|S(x,t) - S(z,\tilde{t})\Big|^p\Big] \leq\; c(p,T)\left(|t-\tilde{t}|^{\frac{1}{4}} + |x-z|^{\frac{1}{2}}\right)^p \tag{6.61}$$

with $c(p,T)$ *independent of* h*. In particular, for* $(x,t),(z,\tilde{t}) \in [0,1]\times[0,T]$ *and some exponent* $\delta < \frac{1}{4}$ *we have the following Hölder regularity estimate*

$$\Big|S(x,t) - S(z,\tilde{t})\Big|^p \leq\; Y(p,T,\delta,h)\left(|t-\tilde{t}|^{\frac{1}{4}-\delta} + |x-z|^{\frac{1}{2}-\delta}\right) \tag{6.62}$$

where $Y(p,T,\delta,h)$ *is a random variable in* L^p *with moment bound independent of* h*:*

$$\mathbb{E}\Big[Y(p,T,\delta,h)^{\frac{1}{\delta}}\Big] \;\leq\; C(p,T,\delta) \tag{6.63}$$

Proof of Lemma 6.4.3. To keep the proof well-arranged, we look at variations in the space and time variable separately. By definition,with $z = x+\xi$,

$$\begin{aligned}&\Big|S(x+\xi,t) - S(x,s)\Big|^{2p}\\ &= \left|\int_0^t \int_0^1 \Big(g^h_{t-s}(x+\xi,y) \;-\; g^h_{t-s}(x,y)\Big) u^h(y,s)\, W(dy,ds)\right|^{2p}.\end{aligned} \tag{6.64}$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|S(x+\xi,t)-S(y,s)\Big|^{2p}\Big]^{\frac{1}{p}} \\
&\overset{BDG}{\leq} c(p)\mathbb{E}\Big[\Big|\int_0^t\int_0^1\Big(\Big(g^h_{t-s}(x+\xi,y)\;-\;g^h_{t-s}(x,y)\Big)\,u^h(y,s)\Big)^2\,dyds\Big|^p\Big]^{\frac{1}{p}} \\
&\overset{\text{Minkowski}}{\leq} \int_0^t\int_0^1\Big|g^h_{t-s}(x+\xi,y)\;-\;g^h_{t-s}(x,y)\Big|^{2p}\mathbb{E}\Big[u^h(y,s)^{2p}\Big]^{\frac{1}{p}}\,dyds \\
&\overset{(6.54)}{\leq} c(\gamma)|x-z|\sup_{[0,1]\times[0,T]}\mathbb{E}\Big[u^h(y,s)^{2p}\Big]^{\frac{1}{p}} \\
&\overset{(6.59)}{\leq} c(\gamma)|x-z|
\end{aligned}
\tag{6.65}
$$

so by taking the $p/2$th power we get

$$
\mathbb{E}\Big[\Big|S(x+\xi,t)-S(y,s)\Big|^{p}\Big]\leq c(\gamma)|x-z|^{p/2} \tag{6.66}
$$

which is the first part of (6.61). Similarly, for the variation in time, with $\tilde{t}=t+r$,

$$
\begin{aligned}
&\Big|S(x,t+r)-S(x,t)\Big|^{2p} \\
&\overset{BDG}{\leq} c(p)\mathbb{E}\Big[\Big|\int_0^{t+r}\int_0^1\Big(\Big(g^h_{t+r-s}(x,y)\;-\;g^h_{t-s}(x,y)\Big)\,u^h(y,s)\Big)^2\,dyds\Big|^p\Big]^{\frac{1}{p}} \\
&\leq \mathbb{E}\Big[\Big|\int_0^{t+r}\int_0^1|g^h_{t+r-s}(x,y)\;-\;g^h_{t-s}(x,y)|^2\,|u^h(y,s)|^2\,dyds\Big|^p.\Big]^{\frac{1}{p}}
\end{aligned}
\tag{6.67}
$$

Therefore, for the integral from zero to t,

$$
\begin{aligned}
&\mathbb{E}\Big[\Big|\int_0^t\int_0^1\Big|g^h_{t+r-s}(x,y)\;-\;g^h_{t-s}(x,y)\Big|^2\,|u^h(y,s)|^2\,dyds\Big|^p\Big]^{\frac{1}{p}} \\
&\qquad\leq\int_0^t\int_0^1\Big|g^h_{t+r-s}(x,y)\;-\;g^h_{t-s}(x,y)\Big|^2\,\mathbb{E}\Big[|u^h(y,s)|^{2p}\Big]^{\frac{1}{p}}\,dyds \\
&\overset{(6.50)}{\leq} c(\gamma,t)\sqrt{|t-\tilde{t}|}\sup_{[0,1]\times[0,T]}\mathbb{E}\Big[|u^h(y,s)|^{2p}\Big]^{\frac{1}{p}}
\end{aligned}
\tag{6.68}
$$

and for the integral from t to $\tilde{t}$,

$$\begin{aligned}
&\mathbb{E}\Big[\Big|\int_t^{t+r}\int_0^1 \Big|g^h_{t+r-s}(x,y)\Big|^2\,|u^h(y,s)|^2\,dyds\Big|^p\Big]^{\frac{1}{p}}\\
&\leq \int_t^{t+r}\int_0^1 \Big|g^h_{t+r-s}(x,y)\,-\,g^h_{t-s}(x,y)\Big|^2\,\mathbb{E}\Big[|u^h(y,s)|^{2p}\Big]^{\frac{1}{p}}\,dyds\\
&\overset{(6.50)}{\leq}\; c(\gamma,t)\sqrt{|t-\tilde{t}|}\sup_{[0,1]\times[0,T]}\,\mathbb{E}\Big[|u^h(y,s)|^{2p}\Big]^{\frac{1}{p}}\\
&\overset{(6.59)}{\leq}\; c(\gamma,t)\sqrt{|t-\tilde{t}|}.
\end{aligned} \tag{6.69}$$

Taking the $p/2$th power gives the second part of (6.61), which concludes the proof. The estimate (6.62) follows from (6.61) by direct application of the Kolmogorov-Centsov theorem. □

Lemma 6.4.4. *Let $g^h_{t-s}(x,y)$ be defined as in* (6.29)*. Then the random variable*

$$B^h(x,t) = \int_0^t\int_0^1 g^h_{t-s}(x,y)\,W(dy,ds) \tag{6.70}$$

is continuous on $[0,1]\times\mathbb{R}^+$*.*
Moreover, for $T>0$ and some exponent $\delta<\frac{1}{4}$ there exists a random variable $Y^h(p,T,\delta)$ in L^p such that for all (x,t) and $(z,\tilde{t})$ in $[0,1]\times[0,T]$ the following inequality holds

$$\Big|B(x,t)-B(z,\tilde{t})\Big| \leq\; Y^h(p,T,\delta)\left(|t-\tilde{t}|^{\frac{1}{4}-\delta}+|x-z|^{\frac{1}{2}-\delta}\right) \tag{6.71}$$

where

$$\sup_h \mathbb{E}\Big[Y_h^p\Big] \;\leq\; C(p,T,\delta) \tag{6.72}$$

with a constant which is independent of h. Moreover, we have for all $1\leq p<\infty$:

$$\sup_h \mathbb{E}\Big[\sup_{[0,1]\times[0,T]}\Big|\int_0^t\int_0^1 g^h_{t-s}(x,y)\,W(dy,ds)\Big|^p\Big] \;\leq C(p,\gamma,T). \tag{6.73}$$

Proof of Lemma 6.4.4. The Hölder continuity statement (6.71) follows directly from estimate (6.62) of lemma 6.4.3. For the second statement (6.73), note first that for small $p\leq 24<q$

$$\mathbb{E}\Big[\sup_{[0,1]\times[0,T]}\big|B^h(x,t)\big|^p\Big] \leq \mathbb{E}\Big[\sup_{[0,1]\times[0,T]}\big|B^h(x,t)\big|^q\Big]^{\frac{p}{q}} \tag{6.74}$$

so it is sufficient to prove the statement for $p > 24$. To this aim, we discretize time into m intervals of size $\frac{T}{m}$ and call the time nodes $t_i := \frac{iT}{m}$. We choose for convenience the spatial grid point $z = \frac{1}{2}$ in the time-discrete stochastic integral and calculate

$$\begin{aligned}
&\sup_{[0,1]\times[0,T]} \left| B^h(x,t) \right| \\
&\qquad \leq \max_{i=1,\dots,m} \left| B^h\left(\frac{1}{2}, t_i\right) \right| \\
&\qquad + \sup_{x\in[0,1]} \sup_{t\in[t_{i-1},t_i]} \left| B^h(x,t) - B^h\left(\frac{1}{2}, t_i\right) \right| \\
&\qquad \overset{(6.71)}{\leq} \max_{i=1,\dots,m} \left| B^h\left(\frac{1}{2}, t_i\right) \right| + Y^h \left(\left(\frac{1}{2}\right)^{\frac{1}{2}-\delta} + \left(\frac{T}{m}\right)^{\frac{1}{4}-\delta} \right) \\
&\qquad \leq \max_{i=1,\dots,m} \left| B^h\left(\frac{1}{2}, t_i\right) \right| + c(p,T,\delta) Y^h.
\end{aligned} \tag{6.75}$$

Furthermore, we estimate

$$\begin{aligned}
\mathbb{E}\Big[\left| B^h\left(\frac{1}{2}, t_i\right) \right|^p \Big] &\overset{BDG}{\leq} C_p \Big[\int_0^t \int_0^1 g_{t-s}^h \left(\frac{1}{2}, y\right)^2 \, dyds \Big]^{\frac{p}{2}} \\
&\overset{(6.50)}{\leq} C_p \left(\frac{3}{\sqrt{8\gamma}} \right)^{\frac{p}{2}} \cdot t^{p/4} \\
&\leq c(p,\gamma,t).
\end{aligned} \tag{6.76}$$

As m is finite, we conclude

$$\begin{aligned}
&\sup_h \mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \left| B^h(x,t) \right|^p \Big] \\
&\quad \leq \sup_h \left(2^p \mathbb{E}\Big[\max_{i=1,\dots,m} \left| B^h\left(\frac{1}{2}, t_i\right) \right|^p \Big] + c(p,T,\delta) \mathbb{E}\Big[Y_h^p \Big] \right) \\
&\quad \overset{(6.72)}{\leq} 2^p \sup_h \sum_{i=1}^m \mathbb{E}\Big[\left| B^h\left(\frac{1}{2}, t_i\right) \right|^p \Big] + c(p,T,\delta) \\
&\quad \overset{(6.76)}{\leq} c(p,\gamma,\delta,T).
\end{aligned} \tag{6.77}$$

□

6.5 Mild solutions

Let $D \subset \mathbb{R}$ be a bounded interval and $x \in D$. Given any bounded continuous initial condition $\phi_0 : \mathbb{R} \to \mathbb{R}$, there exists a unique solution

u to the heat equation

$$\begin{aligned} \partial_t \Phi &= \gamma \Delta \Phi \qquad \text{in } D \times \mathbb{R}^+ \\ \Phi(x,0) &= \phi_0(x) \qquad \text{on } D \times \{0\} \\ &+ B.C. \end{aligned} \tag{6.78}$$

which is given by the formula

$$\Phi(x,t) = \int_D g_t(x,y)\phi_0(y)\,dy \tag{6.79}$$

where

$$g_t(x,y) = \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{|x-y|^2}{4t}\right) \tag{6.80}$$

is called the Green's function or fundamental solution of the heat equation. We define the associated Green's operator as

$$(G_t\phi_0)(x) \ = g_t\phi_0(x) \ = \int_D g_t(x,y)\phi_0(y)\,dy \tag{6.81}$$

which allows us to write the solution to the heat equation simply as $\Phi(\cdot,t) = G_t\phi_0$. The above facts motivate the following definition:

Definition 6.5.1. A random field u is called *mild solution* to the equation (6.2) if (i) u is almost surely continuous, measurable, and (ii) satisfies for all $(x,t) \in D \times \mathbb{R}^+$

$$\begin{aligned} u(x,t) \ = \ \int_D g_t(x,y)u_0(y)\,dy \ &- \int_0^t \int_D g_{t-s}(x,y)V'(u(y,s))\,dyds \\ &+ \sqrt{2\sigma}\int_0^t \int_D g_{t-s}(x,y)\,W(dy,ds) \end{aligned} \tag{6.82}$$

Let $D \subset \mathbb{R}$ be a bounded interval and $x \in D$. Given any bounded continuous initial condition u_0, the heat kernel can be written as

$$g_t(x,y) \ = \ \sum_{k=1}^{\infty} e^{-\lambda_k t} v_k(x) v_k(y) \tag{6.83}$$

where $\lambda_k = \gamma\pi^2 k^2$ are the eigenvalues and $v_k(x) = \sin(\pi k x)$ the eigenfunctions of the Laplacian with corresponding boundary conditions. We recall the definition of a mild solution to an SPDE in the sense of Walsh [116] for the convenience of the reader:

Definition 6.5.2 (mild solution)**.** A random field u is called *mild solution* to the equation (6.2) if (i) u is almost surely continuous, measurable, and (ii) satisfies for all $(x,t) \in D \times \mathbb{R}^+$

$$\begin{aligned} u(x,t) \;=\; \int_D g_t(x,y) u_0(y)\, dy \;-\; \int_0^t \int_D g_{t-s}(x,y) V'(u(y,s))\, dyds \\ +\sqrt{2\sigma} \int_0^t \int_D g_{t-s}(x,y)\, W(dy,ds). \end{aligned} \tag{6.84}$$

We recall the following existence result by Gyöngy and Pardoux [68].

Proposition 6.5.3. *[Existence of mild solutions of the Stochastic Allen-Cahn equation] For every initial conditions $u_0 \in C([0,1])$, the SPDE* (6.2) *admits a unique mild solution. Moreover, for all times $T > 0$ and $p \geq 1$,*

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u(x,t)|^p \right] \;\leq\; C(T,p). \tag{6.85}$$

The random field u is 2α-Hölder in space and α-Hölder in time for every $\alpha \in \left(0, \frac{1}{4}\right)$.

The following existence result on mild solutions of the system of SDEs (6.3) is derived by classical SDE methods (see e.g. [74]), using the one-sided bound $-V(u)\cdot u \leq C(h)\left(1 + \|u\|_{L^2}^2\right)$ for fixed h:

Proposition 6.5.4. *[Existence of mild solutions of the discrete system] Given a suitable deterministic initial condition $u_0 \in C^4([0,1])$ with piecewise linear approximation $u^h(0)$, the rescaled system of SDEs* (6.3) *admits a unique mild solution for all times T.*

6.6 A priori estimates

In this section, we prove a bound on the moments of our discrete solution, which is independent of the grid size h. As the nonlinearity V' is not lipschitz continuous, we will first truncate the discrete solution and prove a moment bound on the truncated solutions u^h_{trunc}. Using a comparison principle, we can then control the discrete solutions and infer a moment bound on the nontruncated discrete solutions.

6.6.1 Uniform bound on moments of truncated solutions

We start with the bound on the moments of truncated solutions, which is separated in the $\sup \mathbb{E}$ and $\mathbb{E} \sup$-bounds part.
Define the truncated drift

$$V'_{trunc}(u) = V'(u)\mathbf{1}_{[-Z,Z]} + V'(Z)\mathbf{1}_{[Z,\infty)} + V'(-Z)\mathbf{1}_{(-\infty,-Z)} \tag{6.86}$$

which is a bounded and globally Lipschitz function. In particular,

$$V'_{trunc}(u) \leq V'(Z) =: M. \tag{6.87}$$

Equations with truncated drift A mild solution to (6.2) with nonlinearity V replaced by V'_{trunc} will be denoted by u_Z and reads

$$\begin{aligned} u_Z(x,t) = & \int_o^1 g_t(x,y)u_0(y)\,dy \\ & - \int_0^t \int_0^1 g_{t-s}(x,y)V'_{trunc}(u(y,s))\,dyds \\ & + \sqrt{2\sigma} \int_0^t \int_0^1 g_{t-s}(x,y)\,W(dy,ds). \end{aligned} \tag{6.88}$$

Similarly, a mild solution to (6.5) with nonlinearity V replaced by V'_{trunc} will be denoted by u^h_{trunc} and reads

$$\begin{aligned} u^h_{trunc}(x,t) = & \int_0^1 g^h_t(x,y)u_0(y)\,dy \\ & - \int_0^t \int_0^1 g^h_{t-s}(x,y)V'_{trunc}(u(y,s))\,dyds \\ & + \sqrt{2\sigma} \int_0^t \int_0^1 g^h_{t-s}(x,y)\,W(dy,ds). \end{aligned} \tag{6.89}$$

Remark 6.6.1. Note that (6.89) differs from (6.5) only by the truncation of the drift term. In particular, for all times $t \leq \tau_Z^h$,

$$u^h(x,t) = u^h_{trunc}(x,t) \qquad \forall x \in [0,1]$$

where we defined the stopping time

$$\tau_Z^h = \inf_{t\in[0,T]}\{\|u^h_{trunc}\|_\infty > Z\} \;=\; \inf_{t\in[0,T]}\{\exists x : |u^h_{trunc}(x,t)| > Z\}. \tag{6.90}$$

It is convenient to write

$$u^h_{trunc} = v^h(x,t) - w^h_Z(x,t) \tag{6.91}$$

where

$$v^h(x,t) \;=\; \int_0^1 g_t^h(x,y)u_0(y)\,dy \tag{6.92}$$

is the solution to the homogeneous problem and

$$\begin{aligned} w_Z^h(x,t) \;&=\; \int_0^t \int_0^1 g^h_{t-s}(x,y)V'_{trunc}(u(y,s))\,dyds \\ &+\; \sqrt{2\sigma}\int_0^t\int_0^1 g^h_{t-s}(x,y)\,W(dy,ds) \\ &\overset{(6.70)}{=} \int_0^t\int_0^1 g^h_{t-s}(x,y)V'_{trunc}(u(y,s))\,dyds \;+\sqrt{2\sigma}B^h(x,t). \end{aligned} \tag{6.93}$$

Lemma 6.6.2. *Let u_0^h be the piecewise linear approximation of the initial data u_0. Let u^h_{trunc} be the solution to the system of SDEs with truncated nonlinearity V'_{trunc} as defined in* (6.89)*. Then, for all times T and all $p > 1$ there exists a constant $C(p,\gamma,T,Z)$ independently of h, such that*

$$\sup_{[0,T]\times[0,1]} \mathbb{E}\left[|u^h_{trunc}|^p\right] \leq C(\gamma,T,p,Z). \tag{6.94}$$

Proof of Lemma 6.6.2. Notice first that the solution to the heat equation is globally bounded: $\|v(t)\|_{L^2} \leq e^{-\lambda_{min}t}\|u_0\|_{L^2}$, where λ_{min} is the smallest eigenvalue under Dirichlet boundary conditions. Similarly, the discrete homogeneous solution can be estimated as

$$\sup_{1\leq i\leq \frac{1}{h}} |v^h(x_i,t)| \;=\; \sup_{1\leq i\leq\frac{1}{h}} \left|(e^{-\gamma A_R^h}u_0^h)_i\right| \;=\; \left\|e^{-\gamma A_R^h}\right\|_\infty \sup_{1\leq i\leq\frac{1}{h}} |u_0^h| \tag{6.95}$$

where $\|.\|_\infty$ denotes the operator norm. Recall that A^h is the coefficient matrix which contains $J(j)$ and γ is the diffusion constant. As A^h is positive definite, its exponential has eigenvalues bounded by one, and therefore $\left\|e^{-\gamma A^h}\right\|_\infty \leq 1$, which leads to

$$\sup_{x\in[0,1]} |v^h(x,t)| \overset{(6.95)}{\leq} \sup_{1\leq i\leq \frac{1}{h}} |u_0(x_i)| \leq \sup_{x\in[0,1]} |u_0| \qquad \forall t>0. \tag{6.96}$$

For the second term, $w_Z^h(x,t)$, we have, for all $t<T$,

$$\begin{aligned} 2^{-p}\mathbb{E}\left[|w_Z^h(x,t)|^p\right] \leq\ & \mathbb{E}\left[\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)V'_{trunc}(u(y,s))\,dyds\right|^p\right] \\ &+\sqrt{2\sigma}\mathbb{E}\left[\left|B^h(x,t)\right|^p\right]. \end{aligned} \tag{6.97}$$

As V'_{trunc} is bounded by M (see (6.87)), we can apply Cauchy-Schwarz and then use the estimate (6.50) on the discrete semigroup to the first term on the right hand side

$$\begin{aligned} \mathbb{E}\Big[\Big|\int_0^t\int_0^1 & g_{t-s}^h(x,y)V'_{trunc}(u(y,s))\,dyds\Big|\Big] \\ &\leq M^p\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)\,dyds\right|^p \\ &\leq M^pT^{p/2}\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)\,dyds\right|^{p/2} \\ &\leq C(\gamma,T,Z). \end{aligned} \tag{6.98}$$

For the stochastic integral, we apply the Burkholder-Davies-Gundy inequality and (6.50) to get

$$\begin{aligned} \mathbb{E}\left[\left|B^h(x,t)\right|^p\right] &\leq C_p\mathbb{E}\left[\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)^2dyds\right|^{p/2}\right] \\ &\leq C(p,\gamma,T,Z), \end{aligned} \tag{6.99}$$

which concludes the proof. □

Lemma 6.6.3. *Let u_0^h be the piecewise linear approximation of the initial data u_0. Let u_{trunc}^h be the solution to the system of SDEs with truncated drift* (6.89). *Then, for all times T and all $p>1$ there exists a constant $C=C(p,\gamma,T,Z)$ independently of h, such that*

$$\sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u_{trunc}^h|^p\right] \leq C. \tag{6.100}$$

Proof of Lemma 6.6.3. Note first that for small $p \leq 24 < q$

$$\mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \left|u^h_{trunc}\right|^p\Big] \leq \mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \left|u^h_{trunc}\right|^q\Big]^{\frac{p}{q}}, \tag{6.101}$$

so it is sufficient to prove the statement for $p > 24$.
Due to (6.91), we can write

$$\left|u^h_{trunc}\right|^p \leq 2^p \left(|v^h(x,t)|^p + |w^h_Z(x,t)|^p\right), \tag{6.102}$$

therefore

$$\begin{aligned}\mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \left|u^h_{trunc}\right|^p\Big] \ &\leq\ 2^p\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |v^h(x,t)|^p\right] \\ &\quad + 2^p\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |w^h_Z(x,t)|^p\right].\end{aligned} \tag{6.103}$$

As seen in (6.96), we have for all $t > 0$

$$\sup_{x\in[0,1]} |v^h(x,t)| \leq \sup_{x\in[0,1]} |u_0(x)|, \tag{6.104}$$

and consequently we get

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |v^h(x,t)|^p\right] \leq \mathbb{E}\left[\sup_{x\in[0,1]} |u_0(x)|^p\right] \leq c. \tag{6.105}$$

Therefore, it remains to bound

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |w^h_Z(x,t)|^p\right] \ \leq\ 2^p((I) + (II)) \tag{6.106}$$

with

$$(I) = \mathbb{E}\left[\sup_{[0,T]\times[0,1]} \left|\int_0^t \int_0^1 g^h_{t-s}(x,y) V'_{trunc}(u(y,s))\, dyds\right|^p\right] \tag{6.107}$$

and $(II) = \sqrt{2\sigma}\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |B^h(x,t)|^p\right]$ with $B^h(x,t)$ as defined in (6.70). First, we estimate with Cauchy-Schwarz

$$\begin{aligned}(I) \ &\leq\ M^p\mathbb{E}\left[\sup_{[0,T]\times[0,1]} \left|\int_0^t \int_0^1 g^h_{t-s}(x,y)\, dyds\right|^p\right] \\ &\leq\ M^pT^{p/2}\mathbb{E}\left[\sup_{[0,T]\times[0,1]} \left|\int_0^t \int_0^1 (g^h_{t-s}(x,y))^2\, dyds\right|^{p/2}\right] \\ &\overset{(6.50)}{\leq}\ c(\gamma)M^pT^{p/4}.\end{aligned} \tag{6.108}$$

Finally, we apply lemma 6.4.4 to (II) and get

$$(II) = \sqrt{2\sigma}\mathbb{E}\left[\sup_{[0,T]\times[0,1]} \left|B^h(x,t)\right|^p\right] \leq c(p,\gamma,t). \tag{6.109}$$

We conclude from (6.105), (6.108) and (6.109)

$$\mathbb{E}\Big[\sup_{[0,1]\times[0,T]} \left|u^h_{trunc}\right|^p\Big] \;\leq\; c(p,\gamma,T,Z). \tag{6.110}$$

As the constant is independent of h, (6.100) follows. □

6.6.2 Uniform moment bound without truncation

To derive the uniform moment bound for the solutions of the original (untruncated) equations from Lemma 6.6.2 and Lemma 6.6.3, we use the following comparison result due to Geiss and Manthey [59].

Proposition 6.6.4. *Let u_1 be the solution to*

$$\begin{aligned} u_1^h(x,t) \;=\; & \int_0^1 g_t^h(x,y)u_0^h(y)\,dy \\ & - \int_0^t\int_0^1 g_{t-s}^h(x,y)V_1((u_1^h(y,s))\,dyds \\ & + \sqrt{2\sigma}\int_0^t\int_0^1 g_{t-s}^h(x,y)\,W(dy,ds) \qquad \text{on } [0,1]\times\mathbb{R}^+ \end{aligned} \tag{6.111}$$

with initial condition $u_1(x,0)$ and u_2 be the solution to

$$\begin{aligned} u_2^h(x,t) \;=\; & \int_0^1 g_t^h(x,y)u_0^h(y)\,dy \\ & - \int_0^t\int_0^1 g_{t-s}^h(x,y)V_2((u_2^h(y,s))\,dyds \\ & + \sqrt{2\sigma}\int_0^t\int_0^1 g_{t-s}^h(x,y)\,W(dy,ds) \qquad \text{on } [0,1]\times\mathbb{R}^+ \end{aligned} \tag{6.112}$$

with initial condition $u_2(x,0)$, and both equations are subject to the same boundary conditions. Suppose that one of the two verifies existence and uniqueness.
If $V_1 \leq V_2$ holds and the initial conditions satisfy $u_1(x,0) \leq u_2(x,0)$, then, for all t and x,

$$u_1^h(x,t) \leq u_2^h(x,t) \qquad \text{almost surely .} \tag{6.113}$$

To use this proposition, we introduce the one-sided truncations

$$
\begin{aligned}
V_Z^+(u) &= V'(u)\mathbf{1}_{(-\infty,Z]} + V'(Z)\mathbf{1}_{[Z,\infty)} \\
V_Z^-(u) &= V'(u)\mathbf{1}_{[-Z,\infty)} + V'(-Z)\mathbf{1}_{(-\infty,-Z]}
\end{aligned}
\tag{6.114}
$$

and denote by $(u^h_{trunc})^+$ and $(u^h_{trunc})^-$ the mild solutions to the associated truncated problems. Note that for fixed Z,

$$- V_Z^-(u) \leq -V'_{trunc}(u) \leq -V_Z^+(u) \tag{6.115}$$

such as

$$- V_Z^-(u) \leq -V'(u) \leq -V_Z^+(u). \tag{6.116}$$

We see that the same moment bounds hold on $u^{h,+}_{trunc}$ and $u^{h,-}_{trunc}$ via comparison:

Lemma 6.6.5. *Let u_0^h be the piecewise linear approximation of the initial data u_0. Let $u^{h,+}_{trunc}$ and $u^{h,-}_{trunc}$ the mild solutions to the system of SDEs with truncated drift $V_Z^+(u)$ and $V_Z^-(u)$, respectively. Then, for all times T and all $p > 1$ there exists a constant $C(p,\gamma,T,Z)$ independently of h, such that*

$$\sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^{h,+}_{trunc}|^p\right] \leq C(p,\gamma,T,Z) \tag{6.117}$$

and

$$\sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^{h,-}_{trunc}|^p\right] \leq C(p,\gamma,T,Z). \tag{6.118}$$

Proof of Lemma 6.6.5. The proof of this lemma is a re-run of the proof of Lemma 6.6.3. □

Proposition 6.6.6. *Let u_0^h be the piecewise linear approximation of the initial data u_0. Let u^h be the solutions to the system of SDEs* (6.5)*. Then, for all times $T > 0$ and all $p > 1$ there exists a constant $C = C(p,\gamma,T)$ which is independent of h, such that*

$$\sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h(x,t)|^p\right] \leq C. \tag{6.119}$$

Proof of Proposition 6.6.6. We apply the comparison Theorem 6.6.4 for

$$V_Z^-(u) \leq -V'(u) \leq -V_Z^+(u) \tag{6.120}$$

to get for all (x,t)

$$u^{h,-}_{trunc}(x,t) \ \le \ u^{h}_{trunc}(x,t) \ \le \ u^{h,+}_{trunc}(x,t) \tag{6.121}$$

where $u^{h,+}_{trunc}$ and $u^{h,-}_{trunc}$ satisfy truncated moment bounds as proved above. Note that

$$|u^h(x,t)| \le \mathbf{1}_{u^h<0}|u^{h,-}_{trunc}(x,t)| + \mathbf{1}_{u^h>0}|u^{h,+}_{trunc}(x,t)|. \tag{6.122}$$

Therefore,

$$\begin{aligned}\sup_{[0,T]\times[0,1]} &|u^h(x,t)|^p \\ &\le \ 2^p\left(\sup_{[0,T]\times[0,1]}|u^{h,-}_{trunc}(x,t)|^p + \sup_{[0,T]\times[0,1]}|u^{h,+}_{trunc}(x,t)|^p\right)\end{aligned} \tag{6.123}$$

which implies in particular

$$\sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]}|u^h(x,t)|^p\right] \le C \tag{6.124}$$

which is what we needed to show. □

6.7 Convergence of solutions

As above in the truncated case, we write $u^h(x,t) = v^h(x,t) - w^h(x,t)$ where $v^h(x,t)$ as in (6.92) and $w^h(x,t) = u^h(x,t) - v^h(x,t)$, i.e.

$$\begin{aligned} w^h(x,t) \;=\; & -\int_0^t \int_0^1 g^h_{t-s}(x,y) V'(u(y,s))\, dyds \\ & + \sqrt{2\sigma} \int_0^t \int_0^1 g^h_{t-s}(x,y)\, W(dy,ds) \end{aligned} \tag{6.125}$$

We start with the proof of convergence of the "homogeneous part" v^h:

Proposition 6.7.1. *Let the initial data u_0 be in C^4. Let $v^h(x,t)$ be the solution to the discrete homogeneous equation as defined in* (6.92)*. Then we have for all $t_0 > 0$ and $\zeta < \frac{1}{2}$.*

$$\sup_{[t_0,T]\times[0,1]} |v^h(x,t) - v(x,t)| \leq c(\gamma, T) h^{2-2\zeta}. \tag{6.126}$$

Proof of Proposition 6.7.1. Note that we can write

$$\begin{aligned} v(x,t) \;&=\; \int_0^1 g_t(x,y) u_0(y)\, dy \\ &=\; u_0(x) \;+\; \int_0^t \int_0^1 g_s(x,y) \Delta u_0(y)\, dyds \\ v^h(x,t) \;&=\; \int_0^1 g^h_t(x,y) u^h_0(y)\, dy \\ &=\; u^h_0(x) \;+\; \int_0^t \int_0^1 g^h_s(x,y) A^h_R u_0(y)\, dyds. \end{aligned} \tag{6.127}$$

This means that

$$\begin{aligned} & \sup_{[0,T]\times[0,1]} |v^h(x,t) - v(x,t)| \\ & \qquad \leq \sup_{x\in[0,1]} |u^h_0(x) - u_0(x)| \\ & \qquad + \sup_{[0,T]\times[0,1]} \left| \int_0^t \int_0^1 g^h_s(x,y) \big(A^h_R u_0(y) - \Delta u_0(y) \big)\, dyds \right| \\ & \qquad + \sup_{[0,T]\times[0,1]} \left| \int_0^t \int_0^1 \big(g^h_s(x,y) - g_s(x,y) \big) \Delta u_0(y)\, dyds \right|. \end{aligned} \tag{6.128}$$

The first term in (6.128) satisfies for continuous u_0 by linear approximation of the initial data:

$$\sup_{x\in[0,1]} |u_0^h(x) - u_0(x)| \leq c \cdot h. \tag{6.129}$$

By (6.28) and as $u_0 \in C^4$ by assumption, A_R^h is a stencil on a uniform grid with consistency order 2. Using (6.50), we can conclude

$$\sup_{[0,T]\times[0,1]} \left| \int_0^t \int_0^1 g_s^h(x,y)\big(A^h u_0(y) - \Delta u_0(y)\big)\, dyds \right| \leq c(\gamma,T)h^2. \tag{6.130}$$

For the last term, we evoke Proposition 6.3.1 on the convergence of semigroups

$$\begin{aligned} &\sup_{[0,T]\times[0,1]} \left| \int_0^t \int_0^1 \big(g_s^h(x,y) - g_s(x,y)\big)\Delta u_0(y)\, dyds \right| \\ &\leq \; \|\Delta u_0\|_{L^2([0,T]\times[0,1])} \cdot \sup_{x\in[0,1]} \left(\int_0^t \int_0^1 \big(g_s^h(x,y) - g_s(x,y)\big)^2 dyds \right)^{1/2} \\ &\leq c(\gamma,T) \cdot h^{2-2\zeta}. \end{aligned} \tag{6.131}$$

Consequently, (6.129), (6.130) and (6.131) give

$$\sup_{[0,T]\times[0,1]} |v^h(x,t) - v(x,t)| \leq c(\gamma,T)\left(h^{2-2\zeta} + h + h^2\right), \tag{6.132}$$

which proves (6.126). □

6.7.1 Convergence of truncated solutions

Recall the splitting (6.91) where $u_{trunc}^h = v^h(x,t) + w_Z^h(x,t)$. Similarly, we split the solution to the continuous truncated equation (6.88) as

$$u_Z(x,t) = v(x,t) + w_Z(x,t) \tag{6.133}$$

where $v(x,t)$ is the solution to the heat equation and w_Z the nonlinear term and the stochastic integral. After having showed convergence of $v^h(x,t)$ to $v(x,t)$ in Proposition 6.7.1, we now study the convergence of the truncated nonlinear term and the stochastic convolution.

Proposition 6.7.2. *Let u_0^h be the piecewise linear approximation of the initial data u_0. Let u_{trunc}^h be the solution to the system of SDEs with truncated drift (6.89), $v^h(x,t)$ be as in (6.92) and let $w_Z^h(x,t) = u_{trunc}^h - v^h(x,t)$ Then, for all times $T > 0$, $\zeta < \frac{1}{2}$ and all $p > 1$ there* *exists a constant $c(p,\delta,\gamma,T,Z)$ independently of h, and an exponent $\delta=\delta(p)>0$ such that*

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]}|w_Z^h-w_Z|^p\right]^{1/p}\leq ch^{\frac{1}{2}-\delta}. \tag{6.134}$$

Proof of Proposition 6.7.2. Note first that for small $p\leq 8<q$

$$\mathbb{E}\Big[\sup_{[0,1]\times[0,T]}\big|w_Z^h-w_Z\big|^p\Big]\leq\mathbb{E}\Big[\sup_{[0,1]\times[0,T]}\big|w_Z^h-w_Z\big|^q\Big]^{\frac{p}{q}} \tag{6.135}$$

so it is sufficient to prove the statement for $p>8$.
By definition,

$$\begin{aligned}&w_Z^h(x,t)-w_Z(x,t)\\&=\int_0^t\int_0^1 g_{t-s}^h(x,y)V'_{trunc}(u_{trunc}^h(y,s))-g_{t-s}(x,y)V'_{trunc}(u_Z(y,s))\,dyds\\&+\sqrt{2\sigma}\int_0^t\int_0^1 g_{t-s}^h(x,y)-g_{t-s}(x,y)\,W(dy,ds).\end{aligned} \tag{6.136}$$

We have

$$|w_Z^h(x,t)-w_Z(x,t)|^p\ \leq\ 2^p\Big((I)+(II)\Big) \tag{6.137}$$

with

$$\begin{aligned}(I)&:=\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)V'_{trunc}(u^h(y,s))-g_{t-s}(x,y)V'_{trunc}(u_Z(y,s))\,dyds\right|^p\\(II)&:=(2\sigma)^{p/2}\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)-g_{t-s}(x,y)\,W(dy,ds)\right|^p.\end{aligned} \tag{6.138}$$

We first estimate term (I), which we again split in two parts:

Step 1: Estimates on term (I) Note that by (6.87)

$$\left|\int_0^t\int_0^1|V'_{trunc}(u_Z(y,s))|^2\,dyds\right|^{p/2}\leq\left|\int_0^t M^2ds\right|^{p/2}=M^pt^{p/2}. \tag{6.139}$$

We split

$$\begin{aligned}(I)\leq 2^p\Big(&\left|\int_0^t\int_0^1\Big(g_{t-s}^h(x,y)-g_{t-s}(x,y)\Big)V'_{trunc}(u_Z(y,s))\,dyds\right|^p\\&+\left|\int_0^t\int_0^1 g_{t-s}^h(x,y)\Big(V'_{trunc}(u_{trunc}^h(y,s))-V'_{trunc}(u_Z(y,s))\Big)\,dyds\right|^p\Big)\\&=2^p((Ia)+(Ib)).\end{aligned} \tag{6.140}$$

For (Ia), we use Cauchy-Schwarz and employ Proposition 6.3.1

$$
\begin{aligned}
(Ia) &= \left| \int_0^t \int_0^1 \left(g_{t-s}^h(x,y) - g_{t-s}(x,y) \right) V'_{trunc}(u_Z(y,s))\, dyds \right|^p \\
&\leq \left(\int_0^t \int_0^1 \left| g_{t-s}^h(x,y) - g_{t-s}(x,y) \right|^2 \, dyds \right)^{p/2} \\
&\times \left(\int_0^t \int_0^1 |V'_{trunc}(u_Z(y,s))|^2 \, dyds \right)^{p/2} \\
&\overset{(6.139)}{\leq} C(p,\gamma,t,lip(V'_{trunc}))h^{2p(1-\zeta)}.
\end{aligned}
\tag{6.141}
$$

For (Ib) we estimate, using Lipschitz continuity of V'_{trunc},

$$
\begin{aligned}
(Ib) &= \left| \int_0^t \int_0^1 g_{t-s}^h(x,y) \Big(V'_{trunc}(u_{trunc}^h(y,s)) - V'_{trunc}(u_Z(y,s)) \Big) \, dyds \right|^p \\
&\leq \left| \int_0^t \int_0^1 (g_{t-s}^h(x,y))^2 \, dyds \right|^{p/2} \\
&\times \left| \int_0^t \int_0^1 \Big(V'_{trunc}(u_{trunc}^h(y,s)) \; - V'_{trunc}(u_Z(y,s)) \Big)^2 \, dyds \right|^{p/2} \\
&\overset{(6.50)}{\leq} C(p,\gamma,t,lip(V'_{trunc})) \; \left(\int_0^t \int_0^1 \left| u_{trunc}^h(y,s) - u_Z(y,s) \right|^2 \, dyds \right)^{p/2}.
\end{aligned}
\tag{6.142}
$$

Using that u_{trunc}^h solves (6.89) and u_Z solves (6.88), we can estimate the RHS in (6.142) further as

$$
\begin{aligned}
&\Big| \int_0^t \int_0^1 \left| u_{trunc}^h(y,s) - u_Z(y,s) \right|^2 \, dyds \Big|^{p/2} \\
&\qquad \leq t^{\frac{p-2}{p}} \int_0^t \int_0^1 \left| u_{trunc}^h(y,s) - u_Z(y,s) \right|^p \, dyds \\
&\qquad \leq 2^p T^{\frac{p-2}{p}} \Big(t \sup_{[0,1]\times[0,T]} |v(y,s) - v^h(y,s)|^p \\
&\qquad + \int_0^t \sup_{[0,1]\times[0,\tilde{s}]} \left| w_Z(y,s) - w_Z^h(y,s) \right|^p \, d\tilde{s} \Big).
\end{aligned}
\tag{6.143}
$$

Summing up estimates (6.141) - (6.143), we get the following estimate for (I):

$$
\begin{aligned}
(I) &\leq 2^p \Big((Ia) + (Ib) \Big) \\
&\lesssim h^{2p(1-\zeta)} + \sup_{[0,1]\times[0,T]} |v(y,s) - v^h(y,s)|^p \\
&+ \int_0^t \sup_{[0,1]\times[0,\tilde{s}]} \left| w_Z(y,s) - w_Z^h(y,s) \right|^p \, d\tilde{s}
\end{aligned}
\tag{6.144}
$$

where we used the notation $\lesssim$ to avoid writing out the constant $C(p, \gamma, T, lip(V'_{trunc}))$ on the RHS. As the lipschitz constant of V'_{trunc} depends only on Z, we can write $C(p, \gamma, T, lip(V'_{trunc})) = C(p, \gamma, T, Z)$.

Step 2: Estimates on term (II) Recall from (6.138) that

$$
\begin{aligned}
(II) &= (2\sigma)^{p/2} \left| \int_0^t \int_0^1 g_{t-s}^h(x,y) - g_{t-s}(x,y) \, W(dy, ds) \right|^p \\
&= (2\sigma)^{p/2} \left| B^h(x,t) - B(x,t) \right|^p .
\end{aligned}
\tag{6.145}
$$

Due to Lemma 6.4.4 and its continuous counterpart, we can estimate

$$
\left| B^h(x,t) - B(x,t) - B^h(x',t) + B(x',t) \right| \leq Y^h |x - x'|^{\frac{1}{2} - \delta} \tag{6.146}
$$

Therefore, with $\theta = \frac{1}{2} - \delta$

$$
\sup_{x\in[0,1]} \left| B^h(x,t) - B(x,t) \right| \; \leq \max_{1\leq i \leq 1/h} |B_i^h(t) - B(x_i,t)| + Y^h h^\theta . \tag{6.147}
$$

This is independent of t, so we get

$$
\sup_{[0,1]\times[0,T]} \left| B^h(x,t) - B(x,t) \right| \; \leq \max_{1\leq i\leq 1/h} \sup_{t\in[0,T]} |B_i^h(t) - B(x_i,t)| + Y^h h^\theta . \tag{6.148}
$$

Recall that $(II) = (2\sigma)^{p/2} \left| B^h(x,t) - B(x,t) \right|^p$, so we take (6.148) to the p-th power to get

$$
\begin{aligned}
\mathbb{E}\left[\sup_{[0,1]\times[0,T]} (II) \right] \leq \; & 2^p \mathbb{E}\left[\max_{i=1\dots1/h} \sup_{t\in[0,T]} |B_i^h(t) - B(x_i,t)|^p \right] \\
& + (2\sigma)^{p/2} \mathbb{E}\left[Y_h^p \right] h^{\theta p} \\
\leq \; & 2^p \sup_{x\in[0,1]} \underbrace{\mathbb{E}\left[\sup_{t\in[0,T]} |B^h(x_i,t) - B(x_i,t)|^p \right]}_{(*)} \\
& + (2\sigma)^{p/2} \mathbb{E}\left[Y_h^p \right] h^{\theta p} .
\end{aligned}
\tag{6.149}
$$

The advantage of getting the supremum over x out of the expectation is that now we can use BDG to estimate $(*)$:

$$
\begin{aligned}
(*) &= \mathbb{E}\left[\sup_{t\in[0,T]} |B^h(x_i,t) - B(x_i,t)|^p\right] \\
&= \mathbb{E}\left[\sup_{t\in[0,T]} \left|\int_0^t \int_0^1 g^h_{t-s}(x_i,y) - g_{t-s}(x_i,y)\, W(dy,ds)\right|^p\right] \\
&\leq c(p) \left|\int_0^T \int_0^1 \left(g^h_{t-s}(x_i,y) - g_{t-s}(x_i,y)\right)^2 \, dyds\right|^{\frac{p}{2}} \\
&\overset{\text{Prop.6.3.1}}{\leq} \quad c(p,\gamma,t)\, h^{2p(1-\zeta)}.
\end{aligned}
\tag{6.150}
$$

In total, we get (resubstituting $\theta = \frac{1}{2} - \delta$)

$$
\mathbb{E}\left[\sup_{[0,1]\times[0,T]} (II)\right] \leq \; c(p,\gamma,\delta,t)\left(h^{2p(1-\zeta)} + h^{\frac{p}{2}-\delta p}\right). \tag{6.151}
$$

Step 3: Conclusion To sum up, we know with (6.137) that

$$
\mathbb{E}\left[\sup_{[0,1]\times[0,T]} |w^h_Z(x,t) - w_Z(x,t)|^p\right] \leq \mathbb{E}\left[\sup_{[0,1]\times[0,T]} (I) + (II)\right] \tag{6.152}
$$

and

$$
\begin{aligned}
\mathbb{E}\left[\sup_{[0,1]\times[0,T]} (I) + (II)\right] \lesssim \left(h^{2p(1-\zeta)} + h^{\frac{p}{2}-\delta p}\right) \\
+ \mathbb{E}\left[\sup_{[0,1]\times[0,T]} |v(y,s) - v^h(y,s)|^p\right] \\
+ \mathbb{E}\left[\int_0^t \sup_{[0,1]\times[0,\tilde{s}]} \left|w_Z(y,s) - w^h_Z(y,s)\right|^p \, d\tilde{s}\right]
\end{aligned}
\tag{6.153}
$$

where we used the notation $\lesssim$ to avoid writing out the constant $c(p,\gamma,\delta,t)$ on the RHS. Gronwall's Lemma with

$f(t) := \mathbb{E}\left[\sup_{[0,1]\times[0,t]} |w_Z^h(x,t) - w_Z(x,t)|^p\right]$ leads

$$\mathbb{E}\left[\sup_{[0,1]\times[0,T]} |w_Z^h(x,t) - w_Z(x,t)|^p\right] \leq c(p,\gamma,\delta,t)\left(h^{2p(1-\zeta)} + h^{\frac{p}{2}-\delta p}\right) + \mathbb{E}\left[\sup_{[0,1]\times[0,T]} |v(y,s) - v^h(y,s)|^p\right]. \tag{6.154}$$

Proposition 6.7.1 gives

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |v^h(x,t) - v(x,t)|^p\right] \leq c(\gamma,t)h^{p(2-2\zeta)} \tag{6.155}$$

which gives the final estimate

$$\mathbb{E}\left[\sup_{[0,1]\times[0,T]} |w_Z^h(x,t) - w_Z(x,t)|^p\right]^{1/p} \leq c(p,\gamma,\delta,t)\left(h^{\frac{1}{2}-\delta} + h^{2-2\zeta}\right). \tag{6.156}$$

Taking the p-th root leads the desired result. $\square$

Conclusions: Convergence of truncated processes In this section, we draw two conclusions from the above propositions. The first one, Proposition 6.7.3, deals with convergence in $L^p(\Omega, C([0,1]\times[0,T]))$ for truncated solutions. The second one, Proposition 6.7.4, deals with almost sure uniform convergence for truncated solutions.

Proposition 6.7.3. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let u_{trunc}^h be the solution to the system of SDEs with truncated drift* (6.89) *and u_Z the solution to the continuous truncated equation* (6.88)*. Then, for all times $T > 0$ and all $p > 1$ there exists a constant $C(p,\gamma,T,Z)$ independently of h, such that for $\zeta < \frac{1}{2}$*

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z|^p\right]^{1/p} \leq Ch^{\frac{1}{2}-\delta}. \tag{6.157}$$

Proof of Proposition 6.7.3 . Proposition 6.7.1 and 6.7.2 give for $T > 0$

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z|^p\right]^{1/p} \leq C(p,\gamma,T,Z)\left(h^{2(1-\zeta)} + h^{\frac{1}{2}-\delta}.\right) \tag{6.158}$$

As for $\zeta < \frac{1}{2}$, $\min\{\frac{1}{2}-\delta, 2-2\zeta\} = \frac{1}{2}-\delta$, the result follows. $\square$

almost sure convergence of truncated solutions Now we prove almost sure convergence for truncated solutions:

Proposition 6.7.4. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let u_{trunc}^h be the solution to the system of SDEs with truncated drift* (6.89) *and u_Z be the solution to the continuous truncated equation* (6.88)*. Then, for all times $T > 0$ and all $\frac{2}{\sqrt{3}} < Z < \infty$ there exists a random variable $\mathbb{X}_Z$, which is almost surely finite, such that for $\zeta < \frac{1}{2}$ and $\eta < \frac{1}{2} - \delta$*

$$\sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z| \;\leq\; \mathbb{X}_Z h^\eta. \tag{6.159}$$

In particular, u_{trunc}^h converges uniformly to u_Z almost surely.

Proof of Proposition 6.7.4 . Define the random variable

$$\mathbb{X}_Z^h := h^{-\eta} \sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z|. \tag{6.160}$$

Note with the Markov inequality that

$$\mathbb{P}[(\mathbb{X}_Z^h)^p \geq 1] \leq \mathbb{E}[(\mathbb{X}_Z^h)^p]. \tag{6.161}$$

We want to apply Borel-Cantelli, so we need to show that for all p,

$$\sum_{N=1}^{\infty} \mathbb{E}[(\mathbb{X}_Z^h)^p] < \infty. \tag{6.162}$$

We see, denoting $X := \sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z|$, that

$$\mathbb{E}[(\mathbb{X}_Z^h)^p] = \mathbb{E}[|X \cdot h^{-\eta}|^p] = h^{-\eta p}\mathbb{E}[|X|^p] \tag{6.163}$$

so we can use Proposition 6.7.3 to show the desired finiteness of moments of $\mathbb{X}_Z^h$ as long as $\eta < \kappa$ with $\kappa < \frac{1}{2} - \delta$ the rate of convergence in the L^p sense, i.e. the only restriction is that $\mathbb{X}_Z^h$ has to diverge slowlier than the truncated solutions converge in the L^p sense.
For such η, Borel-Cantelli can be applied and gives

$$\begin{aligned} \mathbb{P}\left[\left(\limsup_{h\to 0}\{(\mathbb{X}_Z^h)^p \geq 1\}\right)^c\right] &\leq \mathbb{E}[(\mathbb{X}_Z^h)^p]\mathbb{P}\left[\bigcup_{N\geq 1}\bigcap_{M\geq N}\{(\mathbb{X}_Z^h)^p \leq 1\}\right] \\ &= 1. \end{aligned} \tag{6.164}$$

Therefore, there exists an $N = N(\omega)$ such that for all $M \geq N$, $(\mathbb{X}_Z^h)^p(\omega) \leq 1$, and consequently

$$\begin{aligned} \sup_h (\mathbb{X}_Z^h)^p(\omega) &= \max_{1\leq k\leq N(\omega)} (\mathbb{X}_Z^h)^p + \sup_{k\geq N(\omega)} (\mathbb{X}_Z^h)^p \\ &\leq \max_{1\leq k\leq N(\omega)} (\mathbb{X}_Z^h)^p + 1 < \infty. \end{aligned} \tag{6.165}$$

Therefore

$$\mathbb{X}_Z := \sup_h \mathbb{X}_Z^h \tag{6.166}$$

is a.s. finite. Consequently, (6.159) holds almost surely, and the proof is complete. □

6.7.2 From truncated to non-truncated solutions

Stopping times and sets of uniform boundedness

Recall the discrete stopping time τ_Z^h introduced in (6.90)

$$\tau_Z^h = \inf_t \{\|u_{trunc}^h(t)\|_\infty > Z\} = \inf_t \{\exists x : |u_{trunc}^h(x,t)| > Z\}$$

We define two more objects:
(1) A stopping time indicating the growth of the absolute value of the truncated solution

$$\tau_Z = \inf_t \{\|u_Z(t)\|_\infty > Z\} = \inf_t \{\exists x : |u_Z(x,t)| > Z\}.$$

Note: For such stopping times it always holds that $\tau_Z \leq \tau_{Z+1}$.
(2) A set of uniformly bounded u_Z Let Ω_Z be the set of "good" events, i.e. the events where the stopping time τ_Z has not appeared up to time T:

$$\Omega_Z = \left\{ \tau_{Z-\delta} > T \text{ and } \liminf_{h\to 0} \tau_Z^h > T \right\}. \tag{6.167}$$

Ω_Z can be decomposed as

$$\Omega_Z = \bigcup_{h_0} \Omega_{Z,h_0} \tag{6.168}$$

where

$$\Omega_{Z,h_0} = \bigcap_{h\leq h_0} \left\{ \tau_{Z-\delta} > T \text{ and } \tau_Z^h > T \right\}. \tag{6.169}$$

Note that $\Omega_Z \subset \Omega_{Z+1}$ and that the following lemma holds - which means that "all paths are good paths in the limit".

Lemma 6.7.5. *For the sets Ω_Z defined in* (6.167) *holds, under the conditions of Proposition 6.7.4,*

$$\mathbb{P}[\Omega_Z] \longrightarrow 1 \text{ for } Z \to \infty.$$

Proof of Lemma 6.7.5. We prove the statement on the complement

$$\mathbb{P}[\Omega_Z^c] \longrightarrow 0 \text{ for } Z \to \infty. \tag{6.170}$$

Step 1: As a first step, we show that

$$\liminf_{h\to 0} \tau_Z^h \geq T. \tag{6.171}$$

Recall that under the conditions of Proposition 6.7.4,

$$\sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z| \;\leq\; \mathbb{X}_Z^h h^\eta \tag{6.172}$$

and so for all $h \leq h(M) = \left(\frac{M}{\delta}\right)^\eta$ we get

$$\sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z| \;\leq\; \delta \tag{6.173}$$

and therefore

$$\sup_{[0,T]\times[0,1]} |u_{trunc}^h| = \sup_{[0,T]\times[0,1]} |u_{trunc}^h - u_Z + u_Z| \leq \sup_{[0,T]\times[0,1]} |u_Z| + \delta \tag{6.174}$$

which gives the equality of the events

$$\{\tau_{Z-\delta} \geq T\} \;=\; \{\sup_{[0,T]\times[0,1]} |u_Z| < Z - \delta\} \;=\; \{\sup_{[0,T]\times[0,1]} |u_{trunc}^h| < Z\}. \tag{6.175}$$

We conclude that for any $\omega \in \{\tau_{Z-\delta} \geq T\}$ we have

$$\sup_{[0,T]\times[0,1]} |u_{trunc}^h| < Z,$$

which is nothing else than

$$\tau_Z^h > T \qquad \text{for all } h \leq h(M) = \left(\frac{M}{\delta}\right)^\eta \tag{6.176}$$

which proves (6.171).
Step 2: Proof of equation (6.170) We conclude from (6.171) that for all $M > 0$

$$\{\liminf_{h\to 0} \tau_{Z-\delta}^h < T\} \cap \{\mathbb{X}_Z < M\} \subset \{\tau_{Z-\delta} < T\} \cap \{\mathbb{X}_Z < M\}. \tag{6.177}$$

We have the following decomposition of $\mathbb{P}[\Omega_Z^c]$ for $0 < \delta < 1$ fixed, for all $Z > \frac{2}{\sqrt{3}}$ and $M > 0$:

$$\mathbb{P}[\Omega_Z^c] = \mathbb{P}[\tau_{Z-\delta} \leq T] + \mathbb{P}\left[\liminf_{h\to 0} \tau_Z^h < T; \mathbb{X}_Z < M\right] + \mathbb{P}[\mathbb{X}_Z > M]. \tag{6.178}$$

We estimate the terms arising in (6.178): As $\mathbb{X}_Z$ is almost surely finite by (6.165) from the last lemma, $\mathbb{P}[\mathbb{X}_Z > M] \to 0$ for $M \to \infty$, so the last term in (6.178) vanishes in the large M-limit. Moreover, by definition of the stopping time,

$$\{\tau_{Z-\delta} < T\} \subset \{\sup_{[0,T]\times[0,1]} |u| > Z - \delta\} \tag{6.179}$$

and by Markov

$$\begin{aligned}\mathbb{P}\left[\{\sup_{[0,T]\times[0,1]} |u| > Z-\delta\}\right] &\leq \frac{1}{(Z-\delta)^p}\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u|^p\right] \\ &\overset{(6.85)}{\leq} \frac{C(T,p)}{(Z-\delta)^p}\end{aligned} \tag{6.180}$$

as $Z \to \infty$ as $\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u|^p\right] < C(T,p)$ due to Proposition 6.5.3. With (6.177) we can bound for all $M > 0$

$$\mathbb{P}\left[\liminf_{h\to 0} \tau_Z^h < T; \mathbb{X}_Z < M\right] \leq \mathbb{P}\left[\tau_{Z-\delta} < T; \mathbb{X}_Z < M\right] \leq \mathbb{P}[\tau_{Z-\delta} < T] \tag{6.181}$$

which brings us to the same case as before. We summarize

$$\begin{aligned}\mathbb{P}[\Omega_Z^c] &= \mathbb{P}[\tau_{Z-\delta} \leq T] + \mathbb{P}\left[\liminf_{h\to 0} \tau_Z^h < T; \mathbb{X}_Z < M\right] + \mathbb{P}[\mathbb{X}_Z > M] \\ &\leq 2\frac{C(T,p)}{(Z-\delta)^p} + \mathbb{P}[\mathbb{X}_Z > M]\end{aligned} \tag{6.182}$$

and conclude (6.170) by first taking the limit in M and then in Z.

□

In fact, (6.170) can be quantifies as such: For any given $\epsilon > 0$ we can choose Z such that

$$\mathbb{P}[\Omega_Z^c] \leq \epsilon. \tag{6.183}$$

As the sets Ω_{Z,h_0} are ordered and increase when h_0 is decreases,

$$\mathbb{P}[\Omega_Z^c] = \mathbb{P}\left[\bigcap_{h_0} \Omega_{Z,h_0}^c\right] = \lim_{h_0\to 0} \mathbb{P}\left[\Omega_{Z,h_0}^c\right] \leq \epsilon. \tag{6.184}$$

This estimate will be useful in the proof of convergence in norm, see Theorem 6.7.7 below.

Almost sure convergence of discrete solutions

We have already proved almost sure convergence for truncated solutions in (6.159), which gave a r.v. $\mathbb{X}_Z$. Now have to remove the truncation.

Theorem 6.7.6. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u^h(x,t)$ be the solution to the system of SDEs* (6.5) *and $u(x,t)$ the solution to* (6.2).
Then, for all times $T > 0$, $\zeta < \frac{1}{2}$ and $\eta < \frac{1}{2} - \delta$ there exists a random variable $\mathbb{X}$, which is almost surely finite,

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \;\leq\; \mathbb{X} h^\eta. \tag{6.185}$$

In particular, we have almost surely uniform convergence of u^h to u.

Proof of Theorem 6.7.6. Recall from above the definition

$$\Omega_Z = \bigcup_{h_0} \Omega_{Z,h_0} = \bigcup_{h_0} \left(\bigcap_{h \leq h_0} \left\{ \tau_{Z-\delta} > T \text{ and } \tau_Z^h > T \right\} \right)$$

We know that the sets Ω_Z defined in (6.167) are increasing in the sense of $\Omega_Z \subset \Omega_{Z+1}$ and $\mathbb{P}[\Omega_Z] \longrightarrow 1$ as $Z \to \infty$ by lemma 6.7.5. We conclude that

$$\mathbb{P}\left[\bigcup_{Z \geq 1} \Omega_Z \right] = 1. \tag{6.186}$$

Define now

$$\Omega_{Z,\mathbb{X}_Z} := \Omega_Z \cap \{\mathbb{X}_Z < \infty\}. \tag{6.187}$$

As $\mathbb{X}_Z := \sup_h \mathbb{X}_Z^h$ is a.s. finite, the union of $\Omega_{Z,\mathbb{X}_Z}$ has again full measure:

$$\mathbb{P}\left[\bigcup_{Z \geq 1} \Omega_{Z,\mathbb{X}_Z} \right] = 1. \tag{6.188}$$

On $\Omega_{Z,\mathbb{X}_Z}$ there exists $h_0(\omega)$ such that for all $h \leq h_0(\omega)$ we are in the good regime $\tau_Z^h > T$ and $\tau_{Z-\delta} > T$. In this regime, we have according to Proposition 6.7.4

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \;=\; \sup_{[0,T]\times[0,1]} |u^h_{trunc} - u_Z| \;\leq\; \mathbb{X}_Z h^\eta \tag{6.189}$$

as $\mathbb{X}_Z := \sup_h \mathbb{X}_Z^h$ is finite on $\Omega_{Z,\mathbb{X}_Z}$. Therefore, for sufficiently small h there exists a finite random variable $\widetilde{\mathbb{X}_Z}$ such that

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \leq \widetilde{\mathbb{X}_Z} h^\eta. \tag{6.190}$$

By choosing

$$\mathbb{X} := \widetilde{\mathbb{X}_Z} \text{ on } \Omega_{Z,\mathbb{X}_Z} \setminus \Omega_{Z-1,\mathbb{X}_{Z-1}} \tag{6.191}$$

we nest the inequalities (6.190) to construct an almost surely finite random variable $\mathbb{X}$ on $\bigcup_{Z>1} \Omega_{Z,\mathbb{X}_Z}$. As this union has full measure due to (6.188), we conclude

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \leq \mathbb{X} h^\eta \tag{6.192}$$

which ends the proof. □

Proof of convergence in norm

Theorem 6.7.7. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u^h(x,t)$ be the solution to the system of SDEs* (6.5) *and $u(x,t)$ the solution to* (6.2)*.*
Then, for all times $T > 0$, $\delta > 0$, $p > 1$ and $\zeta < \frac{1}{2}$, there exists a constant $c = c(\gamma, p, T)$ such that

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^p\right]^{1/p} \leq c h^{\frac{1}{2}-\delta}.$$

Proof of Theorem 6.7.7. We split the expectation into two parts

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^p\right] = \mathbb{E}\left[\mathbf{1}_{\Omega_{Z,h_0}} \sup_{[0,T]\times[0,1]} |u^h - u|^p + \mathbf{1}_{\Omega^c_{Z,h_0}} \sup_{[0,T]\times[0,1]} |u^h - u|^p\right]. \tag{6.193}$$

Note that by definition of the set Ω_{Z,h_0}, up to time T the absolute value has not reached the value Z yet, and as the $\sup_{[0,T]\times[0,1]}$-norm goes only up to time T, we can replace u by the truncated solution u_Z on the set Ω_{Z,h_0}.
Consequently, we can write for all $h < h_0$

$$\begin{aligned}(6.193) = {} & \mathbb{E}\left[\mathbf{1}_{\Omega_{Z,h_0}} \sup_{[0,T]\times[0,1]} |u^h_{trunc} - u_Z|^p\right] \\ & + \mathbb{E}\left[\mathbf{1}_{\Omega^c_{Z,h_0}} \sup_{[0,T]\times[0,1]} |u^h - u|^p\right].\end{aligned} \tag{6.194}$$

As by virtue of Proposition 6.6.6 and Proposition 6.5.3

$$\begin{aligned}\sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^{2p}\right] &\le \sup_h \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h(x,t)|^{2p}\right] \\ &\quad + \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u(x,t)|^{2p}\right] \\ &\le C(\gamma, p, T)\end{aligned} \tag{6.195}$$

the second term in (6.194) can be reformulated with Cauchy-Schwarz

$$\begin{aligned}\mathbb{E}\left[\mathbf{1}_{\Omega^c_{Z,h_0}} \sup_{[0,T]\times[0,1]} |u^h - u|^p\right] \;&\le\; \mathbb{P}\left[\Omega^c_{Z,h_0}\right]^{1/2} \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^{2p}\right]^{1/2} \\ &\le\; c(\gamma,p,T)\cdot \mathbb{P}\left[\Omega^c_{Z,h_0}\right]^{1/2}.\end{aligned} \tag{6.196}$$

Employing Proposition 6.7.3, equation (6.194) reads

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^p\right] \overset{(6.157)}{\le} c(\gamma,p,T) h^{\frac{p}{2}-\delta p} + C(p,T)\cdot \mathbb{P}\left[\Omega^c_{Z,h_0}\right]^{1/2}. \tag{6.197}$$

Equation (6.184) implies that we can choose some h_0 such that $\mathbb{P}\left[\Omega^c_{Z,h_0}\right] \le 2\epsilon$, and so

$$\mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^p\right] \;\le\; c(\gamma,p,T) h^{\frac{p}{2}-\delta p} + C(p,T)\sqrt{\epsilon}. \tag{6.198}$$

But ϵ was arbitrary, so we take the p-th root and conclude

$$\limsup_{h\to 0} \mathbb{E}\left[\sup_{[0,T]\times[0,1]} |u^h - u|^p\right] \;\le\; c(\gamma,p,T) h^{\frac{1}{2}-\delta} \tag{6.199}$$

which proves the theorem. □

Chapter 7

Metastable transition times

As pointed out already in Chapter 2 and 6, due to the influence of the noise, arbitrarily small random fluctuations can enable transitions between stable states at large time scales. Whether such transitions are observed depends on the timescale of interest.
The aim of this chapter is to show that the transition times of the interacting particle system with long-range interaction, which we studied in the last chapter, converge as the system size goes to infinity, to the transition times of the limit SPDE. As very precise estimates on the transition times of the limit SPDE have been obtained in the last years, we know therefore the behaviour of the interacting particle system for large system size very well.

7.1 Literature remarks

In the simplest situation when we have exactly one particle X, so no interaction, the law of motion we discussed in Chapter 6 reads

$$dX(t) = -\nabla V(X(t))dt + \sqrt{2\sigma}dB(t). \tag{7.1}$$

We know that for small noise strength, i.e. small σ, solutions to (7.1) typically spend a long time near the local minima of V, and transitions between these local minima happen rather quickly. The time which the solution needs to make this transition is called the *transition time*. The mean transition time between the two minima of the potential V is governed by Kramers law [81]:

Let the evolution of the particle X start in a local minimum X_0 of the potential V. Let τ denote the transition time from X_0 to the global minimum X_{min} via a saddle point z.
Kramers law reads in this situation

$$\mathbb{E}(\tau) = \frac{2\pi}{\sqrt{(-V''(X_0))V''(z)}} e^{|V(z)-V(X_0)|/\varepsilon} \left[1 + O_\varepsilon(1)\right]. \tag{7.2}$$

We call $K = \frac{2\pi}{\sqrt{(-V''(X_0))V''(z)}}$ the prefactor, $\beta = \frac{1}{\varepsilon}$ the inverse temperature and $V(z) - V(X_0)$ the activation energy.
The generalisation of Kramer's law to the multidimensional case is called Eyring-Kramers formula, it is attributed to Eyring.

In this following, we quickly describe the results on Eyring-Kramers formula for systems of SDEs with drift given by a double well term $\frac{1}{4}u^4 - \frac{1}{2}u^2$. We refer to the book [24] for a detailed description.
First results for the nearest-neighbour system were obtained in the framework of large deviation theory. Faris and Jona-Lasinio [51] used the large deviations approach by Freidlin and Wentzell [55] to derive the exponential asymptotics of the transition times. Martinelli, Olivieri and Scoppola [88] analysed the expected value of the time to switch from one energy minimizer to a neighbourhood of the other and proved that, on this timescale, the exit probability is exponentially distributed in time. Note that the large deviation approach does not allow to obtain the prefactor in (7.2). Calculations for the prefactor were done by Maier and Stein in a series of works [85, 86, 112].
In recent years, the metastable behaviour of the finite bistable system with nearest-neighbour interaction, for fixed system size N, was investigated by several authors. We mention here as an example Berglund, Fernandez and Gentz [19], who made a very detailed study including bifurcations and symmetries using the theory of large deviations.

To obtain better estimates than with large deviation theory (which identifies the most likely paths and estimates their probabilities), one has to take another, non-pathwise viewpoint:
The potential theoretic approach deals with *metastable sets* and studies when the solution paths are visiting these sets. Important notions in this framework are therefore the *hitting times* of these metastable sets and the expression of the expected transition time in terms of so-called "Newtonian capacities" between sets. These capacities can be estimated by a variational principle involving Dirichlet forms.

The potential theoretic approach to metastability was initiated by Bovier, Eckhoff, Gayrard and Klein [25], where an Eyring-Kramers type law with sharp asymptotics for these transition times for any fixed dimension d was proved. Their result gave an improved error estimate of $1+O\left(\sqrt{\varepsilon\left[\ln\left(\frac{1}{\varepsilon}\right)\right]^3}\right)$, see [24] and the original article [25] for details.

The generalisation of the result in [25] to a system with nearest-neighbour interaction needs a control of the error terms that are uniform in the dimension of the system, and a result on the convergence of prefactors which was shown in [8]. An Eyring-Kramers-type law for the Stochastic Allen-Cahn equation has been proved in [7], it reads in the simple case of two minima u^- and u^- with $\mathcal{E}(u^-) \geq \mathcal{E}(u^+)$ and one saddle point z as

$$\mathbb{E}(\tau(B_\rho(u^+))) = \frac{2\pi}{|\lambda^-(z)|}\sqrt{\frac{|\mathrm{Det}\mathcal{H}_z\mathcal{E}|}{\mathrm{Det}\mathcal{H}_{u^-}\mathcal{E}}}e^{|\mathcal{E}(z)-\mathcal{E}(u^-)|/\varepsilon}\left[1+\psi(\varepsilon)\right] \tag{7.3}$$

with error term $\psi(\varepsilon) = O\left(\sqrt{\varepsilon}\log(\varepsilon)^{3/2}\right)$. Here, $\mathcal{E}$ denotes the Ginzburg-Landau free energy

$$\mathcal{E}(u) = \int_0^1 \frac{1}{2}\left|\nabla u(x)\right|^2 + W(u(x))\,dx, \tag{7.4}$$

$\mathcal{H}_u\mathcal{E}$ is the Hessian operator of $\mathcal{E}$ and $\lambda^-(z)$ the unique negative eigenvalue of $\mathcal{H}_z\mathcal{E}$. Similar results for a more general class of one-dimensional parabolic stochastic partial differential equations, which include also the bifurcation cases, were obtained in [20].

7.2 Convergence of metastable transition times

In this section we prove the convergence of transition times of our discrete system to the transition times of the stochastic Allen-Cahn equation. For the convenience of the reader, we recall the discrete system with long-range interaction in integral form (6.5) from Chapter 6:

$$\begin{aligned} u^h(x,t) = & \int_0^1 g_t^h(x,y)u_0^h(y)\,dy - \int_0^t\int_0^1 g_{t-s}^h(x,y)V'((u^h(y,s))\,dyds \\ & +\sqrt{2\sigma}\int_0^t\int_0^1 g_{t-s}^h(x,y)\,W(dy,ds). \end{aligned} \tag{7.5}$$

Here, we denoted by $g_t^h(x,y)$ the semigroup associated with the discrete operator $-\gamma A_R^h$ defined in (6.7).
The stochastic Allen-Cahn equation reads

$$\begin{aligned}\partial_t u(x,t) =& \gamma A u(x,t) - V'(u(x,t)) \; + \sqrt{2\sigma}\frac{\partial^2}{\partial_x \partial_t} W(x,t) \\ & \qquad \text{for } (x,t) \in [0,1] \times \mathbb{R}^+ \\ u(0,\cdot) =& u_0,\end{aligned} \tag{7.6}$$

where $A = \frac{1}{2}\Delta$ is the Laplace operator with periodic boundary conditions. Recall that $\gamma > 0$ is the diffusion constant, V a double well potential, $\frac{\partial^2}{\partial_x \partial_t} W(x,t)$ denotes space-time white noise and $\sqrt{2\sigma}$ is the intensity of the noise.
As mentioned above, we know the mean transition time of the stochastic Allen-Cahn equation very well, as an Eyring-Kramers formula (7.3) is available. The convergence theorem of the transition times therefore allows us to gain insight on the transition times of our discrete system for large N.

The proof relies on the result on almost sure convergence of the system of SDEs to the SPDE as proved in the last chapter (Theorem 6.7.6). We recall the statement for the convenience of the reader:

Theorem 7.2.1. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u^h(x,t)$ be the solution to the system of SDEs* (7.5) *and $u(x,t)$ the solution to* (7.6)*.*
Then, for all times $T > 0$, $\zeta < \frac{1}{2}$ and $\eta < \frac{1}{2} - \delta$ there exists a random variable $\mathbb{X}$, which is almost surely finite,

$$\sup_{[0,T]\times[0,1]} |u^h(x,t) - u(x,t)| \;\leq\; \mathbb{X} h^{\eta}. \tag{7.7}$$

In particular, we have almost surely uniform convergence of u^h to u.

Definition of hitting times. Given the initial condition u_0 close to one minimum of the potential V. We fix a function $u_{\min} \in C([0,1])$ close to the other minimum and take a ball of radius ρ around $u_{\min}$. We want to estimate the time that our solution $u(t)$ enters this neighbourhood of $u_{\min}$ for the first time. As a neighbourhood of $u_{\min}$ we take the open ball in $L^q([0,1])$.
We define the discrete hitting time τ^h of the $L^q([0,1])$-ball around $u_{\min}$ as

$$\tau^h(\rho,q) := \inf\left\{t > 0, \|u(t)^h - u^h_{\min}\|_{L^q([0,1])} < \rho\right\}. \tag{7.8}$$

Analogously, the continuous hitting time τ of the $L^q([0,1])$-ball around $u_{\min}$ is defined as

$$\tau(\rho, q) := \inf\left\{t > 0, \|u(t) - u_{\min}\|_{L^q([0,1])} < \rho\right\}. \tag{7.9}$$

In order to prove Theorem 7.2.4, we need the following Lemmata, which make use of Theorem 7.2.1:

Lemma 7.2.2. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u^h(x,t)$ be the solution to the system of SDEs* (7.5) *and $u(x,t)$ the solution to* (7.6)*. Let $R \sim h^{-\zeta}$ with $\zeta < \frac{1}{2}$. Let $\tau^h(\rho, q)$ be the transition time for* (7.5) *as defined in* (7.8) *and $\tau(\rho, q)$ be the transition times of* (7.6) *as in* (7.9)*.*
Then, for all $\delta > 0$, $T > 0$ and $q \in [1, \infty)$,

$$\begin{aligned} \min\{\tau(\rho+\delta, q), T\} &\leq \liminf_{h\to 0} \min\{\tau^h(\rho, q), T\} \\ &\leq \limsup_{h\to 0} \min\{\tau^h(\rho, q), T\} \\ &\leq \min\{\tau(\rho-\delta, q), T\} \end{aligned} \tag{7.10}$$

almost surely.

Proof of Lemma 7.2.2. Note that for $u \in C([0,1]\times[0,T])$,

$$\sup_{t\in[0,T]} \left[\int_0^1 |u(x,t)|^q \, dx\right]^{\frac{1}{q}} \leq \sup_{[0,T]\times[0,1]} |u(x,t)|. \tag{7.11}$$

Therefore, if u^h converges in $L^p(\Omega, C([0,1]\times[0,T]))$, then it converges also in $L^p(\Omega, C([0,T], L^q([0,1])))$. Moreover, Theorem 7.2.1 implies also the almost sure convergence of u^h to u in $C([0,T], L^q([0,1]))$. Consequently, there exists $h_0(\omega)$ such that for all $h \leq h_0(\omega)$

$$\sup_{t\in[0,T]} \left[\int_0^1 |u^h - u|^q \, dx\right]^{\frac{1}{q}} (\omega) \leq \frac{\delta}{2} \tag{7.12}$$

and

$$\|u^h_{\min} - u_{\min}\|_{L^q([0,1])} \leq \frac{\delta}{2}. \tag{7.13}$$

We see that

$$\|u(t) - u_{\min}\|_{L^q([0,1])} \geq \rho + \delta \tag{7.14}$$

and calculate

$$\begin{aligned} \|u(t) - u_{\min}\|_{L^q([0,1])} &\leq \|u(t) - u^h(t)\|_{L^q([0,1])} + \|u^h(t) - u^h_f\|_{L^q([0,1])} \\ &\quad + \|u^h_f(t) - u_{\min}\|_{L^q([0,1])} \\ &\leq \delta + \|u^h(t) - u^h_f\|_{L^q([0,1])} \end{aligned} \tag{7.15}$$

which gives

$$\min\{\tau^h(\rho, q), T\} \geq t \tag{7.16}$$

and

$$\min\{\tau(\rho + \delta, q), T\} \leq \liminf_{h\to 0} \min\{\tau^h(\rho, q), T\}. \tag{7.17}$$

Similarly, we have for $t \leq \min\{\tau^h(\rho, q), T\}$ and all $h \leq h_0(\omega)$

$$\begin{aligned} \|u^h(t) - u_f^h\|_{L^q([0,1])} &\leq \|u(t) - u^h(t)\|_{L^q([0,1])} + \|u(t) - u_{\min}\|_{L^q([0,1])} \\ &+ \|u_f^h(t) - u_{\min}\|_{L^q([0,1])} \\ &\leq \ \delta + \|u(t) - u_{\min}\|_{L^q([0,1])} \end{aligned} \tag{7.18}$$

from which we conclude

$$\limsup_{h\to 0} \min\{\tau^h(\rho, q), T\} \ \leq \ \min\{\tau(\rho - \delta, q), T\} \tag{7.19}$$

which ends the proof. $\square$

Thanks to Lemma 7.2.2, we get the following estimate:

Lemma 7.2.3. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u^h(x,t)$ be the solution to the system of SDEs* (7.5) *and $u(x,t)$ the solution to* (7.6)*. Let $R \sim h^{-\zeta}$ with $\zeta < \frac{1}{2}$. Let $\tau^h(\rho, q)$ be the transition time for* (7.5) *as defined in* (7.8) *and $\tau(\rho, q)$ be the transition times of* (7.6) *as in* (7.9)*.*
Then, for all $\rho > 0$,

$$\tau(\rho, q) \ \leq \ \liminf_{h\to 0} \tau^h(\rho, q) \ \leq \ \limsup_{h\to 0} \tau^h(\rho, q) \ \leq \ \tau(\rho^-, q) \qquad a.s. \tag{7.20}$$

where $\tau(\rho^-, q) = \lim_{\delta\to 0^+} \tau(\rho - \delta, q)$.

Proof of Lemma 7.2.3. By Lemma 7.2.2 and monotone convergence,

$$\tau(\rho + \delta, q) \ \leq \ \liminf_{h\to 0} \tau^h(\rho, q) \ \leq \ \limsup_{h\to 0} \tau^h(\rho, q) \ \leq \ \tau(\rho - \delta, q). \tag{7.21}$$

As $\rho \mapsto \tau(\rho, q)$ and $\rho \mapsto \tau^h(\rho, q)$ are cadlag functions, we can deduce

$$\tau(\rho, q) \ \leq \ \liminf_{h\to 0} \tau^h(\rho, q) \ \leq \ \limsup_{h\to 0} \tau^h(\rho, q) \ \leq \ \tau(\rho^-, q) \qquad a.s. \tag{7.22}$$

which concludes the proof. $\square$

We can now proceed to the proof of the main statement, the convergence of transition times:

Theorem 7.2.4. *Let u_0^h be the piecewise linear approximation of an initial data $u_0 \in C^4$. Let $u^h(x,t)$ be the solution to the system of SDEs* (7.5) *and $u(x,t)$ the solution to* (7.6)*. Let $R \sim h^{-\zeta}$ with $\zeta < \frac{1}{2}$. Let $\tau^h(\rho,q)$ be the transition time for* (7.5) *as defined in* (7.8) *and $\tau(\rho,q)$ be the transition times of* (7.6) *as in* (7.9).*
Then, for almost all $\rho > 0$,

$$\tau^h(\rho,q) \longrightarrow \tau(\rho,q) \qquad \text{a.s. as } h \to 0 \tag{7.23}$$

and

$$\mathbb{E}\left[\tau^h(\rho,q)\right] \longrightarrow \mathbb{E}\left[\tau(\rho,q)\right] \qquad \text{as } h \to 0. \tag{7.24}$$

Proof of Theorem 7.2.4. Lemma 7.2.3 shows that the statement holds for all points of continuity of $\rho \mapsto \tau(\rho,q)$. As $\rho \mapsto \tau(\rho,q)$ is càdlàg and increasing on almost all $\omega \in \Omega$, there are at most countably many points of discontinuity.
Let us denote by $\mathscr{D}$ the points of discontinuity for a fixed ω and by $\mathscr{J}$ the jump set:

$$\mathscr{D}(\omega) = \left\{\rho \in \mathbb{R}^+, \tau(\rho^-,q) \neq \tau(\rho,q)\right\} \qquad \mathscr{J} = \bigcup_\omega \{\omega\} \times \mathscr{D}(\omega). \tag{7.25}$$

As there are at most countably many points of discontinuity for each ω, we immediately conclude that $\mathscr{D}(\omega)$ has Lebesgue measure zero.
Let us now take a different perspective and call $\mathscr{N}$ the set of ω for which τ is discontinuous in ρ. This gives us an alternative description of the jump set in terms of unions of ρ:

$$\mathscr{J} = \bigcup_\omega \{\omega\} \times \mathscr{D}(\omega) \;=\; \bigcup_{\rho \in \mathbb{R}^+_{>0}} \mathscr{N}(\rho) \times \{\rho\}. \tag{7.26}$$

We calculate with Fubini

$$\begin{aligned} \int_{\mathbb{R}^+_{>0}} \mathbb{P}[\mathscr{N}(\rho)] d\rho \;&=\; \int_{\mathbb{R}^+_{>0}} \mathbb{E}[\mathbf{1}_{\mathscr{N}(\rho)}] d\rho \\ &=\; \int_\Omega \int_{\mathbb{R}^+_{>0}} \mathbf{1}_{\mathscr{J}(\omega\rho)} d\rho \, d\mathbb{P}(\omega) \\ &=\; \int_{\Omega\setminus\mathscr{N}} \int_{\mathbb{R}^+_{>0}} \mathbf{1}_{\mathscr{D}(\omega)}(\rho) d\rho \, d\mathbb{P}(\omega) \;=\; 0 \end{aligned} \tag{7.27}$$

as

$$\int_{\mathbb{R}^+_{>0}} \mathbf{1}_{\mathscr{D}(\omega)}(\rho) d\rho \;= 0 \qquad \text{for all } \omega \in \Omega \setminus \mathscr{N}. \tag{7.28}$$

Therefore, there exists a set of measure zero N of $\mathbb{R}^+$ such that

$$\mathbb{P}[\mathscr{N}(\rho)] = 0 \qquad \text{for all } \rho \in \mathbb{R}^+ \setminus N. \tag{7.29}$$

Consequently, we have

$$\tau(\rho^-, q)(\omega) = \tau(\rho, q)(\omega) \qquad \text{for all } \rho \in \mathbb{R}^+ \setminus N \text{ and } \omega \in \Omega \setminus \mathscr{N}(\rho). \tag{7.30}$$

This means that for $h \to 0$

$$\tau^h(\rho, q) \longrightarrow \tau(\rho, q) \qquad \omega - \text{a.s.}, \ \rho - \text{a.e.}. \tag{7.31}$$

By dominated convergence, we conclude that for all $\rho \in \mathbb{R}^+ \setminus \mathscr{D}$

$$\mathbb{E}\left[\tau^h(\rho, q)\right] \longrightarrow \mathbb{E}\left[\tau(\rho, q)\right] \qquad \text{as } h \to 0 \tag{7.32}$$

which is the statement of the theorem. $\square$

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